Double chip waveforms for asynchronous DS-CDMA systems with random signature sequences

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Abstract: In an asynchronous DS-CDMA system using random signature sequences and correlation receivers, the signal-to-interference-plus-noise ratio (SINR) at the output of each correlation receiver depends on the shape of the chip waveform employed. The use of double chip waveforms instead of a single chip waveform is investigated, in order to increase the SINR performance. An analytical expression for the SINR when double chip waveforms are used is obtained. To evaluate the performance of the proposed technique, an extension to Holtzman’s approximation of the error probability for the case of double chip waveforms and signalling over an additive white Gaussian noise channel is derived. Numerous numerical examples are given to demonstrate the advantage of using double chip waveforms over the single chip waveform. Moreover, the advantage of using double chip waveforms for improving the SINR performance over a multipath Rayleigh fading channel is also demonstrated.

1 Introduction

Although in direct-sequence code-division multiple access (DS-CDMA) systems, multiuser detection has been shown to outperform single-user detection (or correlation receiver) [1], the complexity of multiuser detection is usually prohibitive in systems with a large number of users. Therefore, conventional single-user detection is still the only practical solution in many DS-CDMA systems. A common and important performance measure for the correlation receiver is the signal-to-interference plus noise ratio (SINR). In general, one wishes to maximise the SINR, or equivalently minimise the variance of the multiple access interference (MAI) at the output of correlation receiver to improve the users’ performance.

In a binary DS-CDMA system, users transmit the information symbols by modulating their own signature waveforms. Each user’s signature waveform in DS-CDMA is usually constructed by modulating a given chip waveform with the users’ signature sequences. Clearly, the SINR at the output of each correlation receiver depends on both the signature sequences and the shape of chip waveform. Recently, the use of random signature sequences has been widely adopted when analysing the performance of DS-CDMA systems [2–5]. With random signature sequences, the average SINR depends only on the chip pulse shape.

Generally, the chip waveform can be either band-limited or time-limited. Though it seems that band-limited waveforms (or more precisely the truncated versions of them) are more popular in practical systems [6] here, only chip waveforms that are limited to one chip interval are considered. Chip waveforms that are time-limited to one chip interval are popular in CDMA research literature [4, 5, 7–9]. Applying the idea proposed in this paper to include chip waveforms that span multiple chip intervals is a natural extension.

It is important to note that when a single, time-limited (to a chip interval) chip waveform is used and when the delays between the desired user and the interfering users are exact multiples of the chip duration, the chip pulse shape has no effect on the amount of MAI. In such situations the MAI depends only on the cross-correlations of the signature sequences, which can be large if the signature sequences are chosen randomly. However, this MAI can be made equal to zero, albeit for the special case when the delays of the interfering users are an exact odd multiple of the chip duration, regardless of the chosen signature sequences as follows. Instead of using a single chip waveform, have two different chip pulses which are alternatively used to construct the signature waveform. If one chip waveform is chosen to be an odd function and the other an even function about the mid-point of the chip duration then the signals of the interfering users are orthogonal to that of the desired user (for the assumed delays) and hence the MAI is zero.

Motivated by the above observation, in this paper we introduce and evaluate the use of double chip waveforms as a means of reducing MAI in asynchronous DS-CDMA systems.

2 System model

The model for an asynchronous binary DS-CDMA system signalling over an AWGN channel under consideration is similar to the one in [7]. There are K users in the system. The k-th user transmits the following spread-spectrum signal over bandwidth W:

\[ y_k(t) = \sum_{i=-\infty}^{\infty} \sqrt{2P_b} \beta_k(i) x_k(t - iT) \times \cos(2\pi f_c t + \theta_k) \]  

where \( P_b \) is the power of the desired user's signal, \( \beta_k(i) \) is the time-index of the chip waveform, \( x_k(t) \) is the chip waveform, \( f_c \) is the carrier frequency, and \( \theta_k \) is the phase shift.

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In (1), \( P \) is the signal power (the common power assumption is made for simplicity but can be relaxed); \( f_c \) is the carrier frequency; \( \theta_k \) is the phase introduced by the \( k \)th modulator; and \( T \) is the symbol duration. The sequence \( \{b_k(t)\} \) is the binary data sequence of user \( k \), which is modelled as a sequence of independent and identically distributed (IID) random variables such that \( \Pr(b_k(t) = +1) = \Pr(b_k(t) = -1) = 1/2 \). Let \( g_1(t) \) and \( g_2(t) \) be two chip pulses time-limited to \([0, T]\) whose energies are normalised such that \( \int_0^T g_1^2(t)dt = \int_0^T g_2^2(t)dt = T \). The signature waveform of the \( k \)th user has duration \( T = N_Tc \) and is constructed as follows:

\[
s_k(t) = \sum_{i=0}^{M} s_k(2i)g_1(t - 2iT_c) + s_k(2i + 1)g_2(t - (2i + 1)T_c)
\]

where \( 2M \) is the number of chips in interval \([0, T]\) and the signature vector \( s_k = [s_k(0), s_k(1), \ldots, s_k(N-1)] \) is modelled as a vector of IID random variables taking values in \([-1,+1]\). To simplify our analysis, it has been assumed in (2) that the processing gain \( N = T/T_c \) is an even number, i.e. \( N = 2M \). However all the results in this paper are still valid for odd \( N \).

The received signal is

\[
y(t) = \sum_{k=1}^{K} \sum_{i=-\infty}^{\infty} \sqrt{2P} b_k(t) s_k(t - iT - \tau_k) \cos(2\pi f_c t + \phi_k) + n(t)
\]

where \( \tau_k \) and \( \phi_k = \theta_k - 2\pi f_c \tau_k \) are the delay and the overall phase shift of the \( k \)th user, which can be modelled as uniform random variables over \([0, T]\) and \([0, 2\pi]\), respectively. The noise \( n(t) \) is additive white Gaussian noise (AWGN) with two-sided power spectral density of \( N_0/2 \).

Without loss of generality, consider the detection of the first information symbol of the \( k \)th user, i.e. \( b_k(0) \). Also, since only relative delays and phase are important one can set \( \tau_k = 0 \) and \( \phi_k = 0 \) and the delays and phase shifts of all other users are interpreted with reference to the \( k \)th user. Ignoring the double frequency component at \( 2f_c \), the output of the \( k \)th correlation receiver is [8]

\[
Z_k = \int_0^T y(t)s_k(t) \cos(2\pi f_c t)dt
\]

\[
= \frac{P}{2} b_k(0) + \frac{P}{2} \sum_{i=1,i\neq k}^{K} I_{k,i} + n
\]

where \( n \) is a Gaussian random variable with zero mean and variance \( N_0T/4 \) and \( I_{k,i} \) is the interference caused by the \( i \)th user, given by:

\[
I_{k,i} = [b_i(-1)R_{k,i}^a(t_\tau) + b_i(1)R_{k,i}^b(t_\tau)] \cos \phi_i
\]

The functions \( R_{k,i}^a(t) \) and \( R_{k,i}^b(t) \) are the continuous-time partial cross-correlation functions between the \( k \)th and the \( i \)th signature waveforms. These functions were originally introduced in [7, 8] and can be written here as \( R_{k,i}^a(t) = \int_0^T s_k(t) s_i(t + T - t)dt \) and \( R_{k,i}^b(t) = \int_0^T s_k(t) s_i(t - T + t)dt \) for \( 0 \leq T \leq T \). When \( k = i \), then denote \( R_k(t) = R_{k,k}(t) \) and \( \hat{R}_k(t) = \hat{R}_{k,k}(t) \).

3 SINR evaluation and bandwidth consideration

From (4), the SINR for the \( k \)th user can be calculated as \( \text{SINR}_k = [E(Z_k)]^2/\text{var}(Z_k) \). Because of the symmetry involved, it suffices to consider the case \( b_k(0) = 1 \), hence \( [E(Z_k)]^2 = PT^2/2 \). Owing to the independence assumptions among the information symbols, the signature sequences and the noise, it is easy to see that the random variables \( I_{k,i} \) and \( n \) in (4) are zero-mean and uncorrelated, thus:

\[
\text{var}(Z_k) = \frac{P}{2} \sum_{i=1,i\neq k}^{K} \text{var}(I_{k,i}) + \frac{N_0T}{4}
\]

Note that \( I_{k,i} \) depends on the random variables \( b_k = [b_k(-1), b_k(0), b_i(-1), b_i(0), \tau_k, \tau_i, s_k, s_i] \). As usual, these random variables are assumed to be mutually independent, hence the variance of \( I_{k,i} \) can be computed as follows:

\[
\text{var}(I_{k,i}) = E_{s,k} \{ E_{\phi_k} \{ E_{b_i} \{ E_{\tau_k} \{ E_{\tau_i} \{ E_{s_i} \{ E_{b_i} \{ E_{\tau_k} \{ E_{\tau_i} \{ s_k(t) s_i(t') b_k(r) \phi_k(t) \tau_k(t') \tau_i(r) s_i(t) \} \} \} \} \} \} \} \} \}
\]

The third equality in the above expression is a consequence of the following identity, whose justification is provided in the Appendix.

\[
\int_0^T \int_0^T [R_{k,i}^a(t) + \hat{R}_{k,i}^a(t)]dt
\]

\[
= 2 \int_0^T \hat{R}_k(t) \hat{R}_i(t)dt
\]

To further evaluate (7), define the following partial autocorrelations and cross-correlations between the chip waveforms

\[
h_m,n(t) = \int_0^T g_m(t) g_n(t + T_c - \tau - t)dt
\]

\[
h_m,n(t) = \int_0^T g_m(t) g_n(t - T_c - \tau - t)dt
\]

where \( 0 \leq \tau \leq T_c \) and \( m, n = 1 \) or 2. Also denote \( h_{m,n}(\tau) = h_{m,n}(0, \tau) \) and \( h_m(\tau) = h_{m,m}(\tau) \) when \( m = n \). Let \( l = \lfloor \tau/T_c \rfloor \) be the integer part of \( \tau/T_c \) and \( r = \tau - lT_c \), then \( l \) and \( r \) are random variables uniformly distributed over \([0, 1, \ldots, N-1]\) and \([0, T_c]\), respectively. If \( l \) is even, it can be shown that:

\[
\hat{R}_k(t) = \hat{h}_1(r) \sum_{j=0}^{(N-1)/2-1} s_k(2j) s_k(l + 2j)
\]

\[
+ \hat{h}_2(r) \sum_{j=0}^{(N-1)/2-1} s_k(2j + 1) s_k(l + 2j + 1)
\]

\[
+ h_{1,2}(r) \sum_{j=0}^{(N-1)/2-2} s_k(2j + 1) s_k(l + 2j + 2)
\]

\[
+ h_{1,2}(r) \sum_{j=0}^{(N-1)/2-2} s_k(2j) s_k(l + 2j + 2)
\]
Likewise, if \( l \) is odd one has:
\[
\hat{R}_k(\tau) = \hat{h}_{21}(r) \sum_{j=0}^{(N-1)/2} s_j(2j) s_k(l + 2j) \\
+ \hat{h}_1(r) \sum_{j=0}^{(N-1)/2} s_j(2j) s_k(l + 2j + 1) \\
+ \hat{h}_{12}(r) \sum_{j=0}^{(N-1)/2} s_j(2j + 1) s_k(l + 2j + 1) \\
+ \hat{h}_2(r) \sum_{j=0}^{(N-1)/2} s_j(2j + 1) s_k(l + 2j + 2) 
\] (12)

Note that the summations in (11) and (12) do not exist if their upper indexes are negative.

Since the components of vector \( s_k \) are IID random variables, it follows that:
\[
E_k = \{ \hat{R}_k(\tau) \} = \begin{cases} \frac{1}{2} \hat{h}_1(r) + \hat{h}_2(r), & \text{if } l = 0 \\ 0, & \text{otherwise} \end{cases} \quad (13)
\]

Thus the variance of \( I_{k,i} \) in (7) becomes
\[
\text{var}(I_{k,i}) = \frac{N}{4T_c} \int_0^{T_c} [\hat{h}_1(r) + \hat{h}_2(r)]^2 dr 
\] (14)

which is the same for all \( i \) (\( i \neq k \)). Let
\[
I = \frac{1}{T_c} \int_0^{T_c} [\hat{h}_1(r) + \hat{h}_2(r)]^2 dr 
\] (15)

be the interference parameter and \( E_k = PT \) (the energy per symbol), then the SINR is the same for every user and given by:

\[
\text{SINR} = \left[ \frac{2E_k}{N_0} \right]^{-1} + \frac{K - 1}{4N} 
\] (16)

Note that when \( g_d(t) = g_d(t) \), \( I \) is just the normalised mean-squared partial chip correlation defined in [3], [5] and the SINR in (16) agrees with the result given in [3] for the single chip pulse. Furthermore the parameter \( I \) can be written in terms of the Fourier transforms of the chip pulses as follows. For \( m = 1, 2 \), let \( u_m(\tau) = \int_{-\infty}^{\infty} g_m(t) g_m(t-\tau) dt = g_m(\tau) * g_m(-\tau) \) be the auto-correlation of the chip pulse \( g_m(t) \). Then \( u_m(\tau) \) is an even function and confined to \([-T_c, T_c]\). Let \( U_m(f) = \mathcal{F}[u_m(\tau)] \) and \( G_m(f) = \mathcal{F}[g_m(\tau)] \), where \( \mathcal{F}[\cdot] \) denotes the Fourier transform. Note that \( U_m(f) = G_m(f)G^*_m(f) = |G_m(f)|^2 \). Now using the fact that \( \hat{u}_m(\tau) = u_m(\tau) \) for \( 0 \leq \tau \leq T_c \) and applying Parseval’s theorem one has:
\[
I = \frac{1}{T_c} \int_0^{T_c} [\hat{h}_1(\tau) + \hat{h}_2(\tau)]^2 d\tau \\
= \frac{1}{T_c} \int_0^{T_c} [u_1(\tau) + u_2(\tau)]^2 d\tau \\
= \frac{1}{2T_c} \int_{-T_c}^{T_c} [u_1(\tau) + u_2(\tau)]^2 d\tau 
\]

Again, when \( g_d(t) = g_d(t) \), (17) reduces to the normalised integration of the fourth power of the magnitude spectrum of the single chip waveform as shown in [4] and [5].

The SINR in (16) clearly depends on the processing gain \( N \) and on the actual chip waveforms through the interference parameter \( I \). To fairly compare different systems which employ different pairs of chip waveforms, it is necessary to impose an equal symbol rate \((1/T)\) and equal bandwidth constraint \( W \), or at least an equal product \( WT \) among these systems. The bandwidth of a communication system is usually judged based on the power spectral density (PSD) of the transmitted signal. The PSD of the transmitted signal in the model (3) is completely determined by the PSD of the equivalent baseband signal \( \sum_{k=-N}^{N} \sum_{i=-\infty}^{\infty} \sqrt{2} p_k(i) s_k(t - iT - \tau_k) \). This PSD can be shown to be \( P(f) = (P(T_c, \omega) (G_1(f))^2 + |G_2(f)|^2) \). A commonly used bandwidth measurement is the fractional out-of-band power (FOBP) bandwidth [10]. Let \( 0 < \eta < 1 \) be arbitrary; the system is said to have FOBP bandwidth \( W \) at level \( \eta \) if:
\[
\int_{-W}^{W} P(f) df = \frac{1}{2T_c} \int_{-W}^{W} \left( (|G_1(f)|^2 + |G_2(f)|^2) \right) df = 1 - \eta 
\] (18)

Given \( \eta \), the product of \( WT_c \) can be found for any pair of chip waveforms from the above equation. At this point, based on (17) and (18), one may suggest that by choosing a single chip waveform with power spectral density
\[
(G(f))^2 = \frac{1}{2} (|G_1(f)|^2 + |G_2(f)|^2) 
\] (19)

then the SINR performance of any choice of double chip waveforms can be obtained by the corresponding single chip waveform with the same bandwidth. This is not possible in general. Granted one can readily obtain the energy density spectrum as indicated in (19). But to obtain the time waveform \( g_d(t) \), one must also specify the phase spectrum. The single chip waveform \( g_d(t) \) obtained through the inverse Fourier transform of \( G(f) \) now will not be necessarily a time-limited (to the interval \([0, T_c]\)) function as specified.

Note also that, if the restriction on time limitation is lifted, the expression of the interference parameter in (17) does not hold for the single chip waveform obtained via (19). This is because for a chip waveform that spans multiple chip intervals, there is generally a non-zero interchip interference (ICI). It can be shown that the interference parameter for a chip waveform whose support is longer than one chip interval is given by [11],
\[
I = \frac{1}{2T_c} \int_{-\infty}^{\infty} |G(f)|^4 df \\
+ \frac{1}{(K-1)T_c} \sum_{m=0}^{\infty} \left[ \int_{-\infty}^{\infty} \cos(2\pi mfT_c) |G(f)|^2 df \right]^2 
\] (20)

where the second term accounts for the ICI.
Since \( N = WT \), the SIRN in (16) can be rewritten as:

\[
\text{SINR} = \left( \frac{2E_b}{N_0} \right)^{-1} + \frac{K - 1}{4} \frac{1}{W^T}
\]

(21)

Though \( W \) can be cancelled, writing \( W_T \) explicitly in (21) is convenient since the value of \( W_T \) is readily determined from the bandwidth constraint of (18). Thus, when \( T \) and \( W \) are fixed, to maximise the SIRN one has to find the pair of chip waveforms that has smallest value of \( W_T \). While obtaining such an optimal pair can be found in [12], the combinations of several common chip waveforms are presented in Section 5 to illustrate the advantage of using double chip waveforms.

4 Error probability analysis

Though SINR is an important and useful performance parameter, error probability is perhaps the most important performance index in any communications systems. It is therefore important to evaluate the performance of the double chip waveforms in terms of the bit error rate (BER).

The exact calculation of the error probabilities of asynchronous DS-CDMA communications systems is often intractable and computationally difficult due to the complexity of asynchronous CDMA systems. Thus, most previous work on this problem has concerned approximations and bounds [2, 7, 13–16]. Among these contributions, the approximation derived by Holtzman [13] seems very attractive since it is simple but gives good accuracy. This approximation has been widely used [16–20] and is generally referred to as the improved Gaussian approximation (IGA). In this Section, the extension of Holtzman’s approximation is provided to approximate the error probabilities of DS-CDMA systems using double chip waveforms. The following derivation yields a more general result than the one established in [21] for a single chip waveform.

From (4) and (5), the decision statistic at the output of the \( k \)-th correlation receiver can be written as:

\[
W_i = b_i(-1)R_{sk}(\tau_i) + b_i(0)R_{sk}(\tau_i)
\]

(22)

Recall that \( \tau_i = \lceil t_i / T_c \rceil \) and \( r_i = t_i - \lceil t_i / T_c \rceil \). At this point, to simplify the notation, we set \( l_i = l \) and \( r = r_i \). Whenever necessary, the appropriate indexes of \( l \) and \( r \) can be restored.

Now if \( l \) is even, then it can be shown that \( W_i \) can be written as follows:

\[
W_i^e = b_i(-1) \sum_{j=0}^{N/2 - 1} s_i(N + 2j - l - 1)
\]

\[
\times \left[ s_k(2j + 2)h_{1/2}(r) + s_k(2j + 1)h_{1/2}(r) \right] + b_i(0) \sum_{j=0}^{N/2 - 1} s_i(2j - l - 1)
\]

\[
\times \left[ s_k(2j + 2)h_{1/2}(r) + s_k(2j + 1)h_{1/2}(r) \right] + b_i(-1) s_k(0)s_i(N - l - 1)h_{1/2}(r)
\]

\[
+ b_i(0) s_k(N - 1)s_i(N - l - 1)h_{1/2}(r)
\]

\[
+ b_i(-1) \sum_{j=0}^{N/2 - 1} s_i(N + 2j - l)
\]

\[
\times \left[ s_k(2j + 2)h_{1/2}(r) + s_k(2j + 1)h_{1/2}(r) \right] + b_i(0) \sum_{j=0}^{N/2 - 1} s_i(2j - l)
\]

\[
\times \left[ s_k(2j + 2)h_{1/2}(r) + s_k(2j + 1)h_{1/2}(r) \right]
\]

(23)

As in [2], with the motivation of reducing complexity, it is important to consider (23) conditioned on the signature sequence of the \( k \)-th user and the random variable \( l \), i.e. \( s_k(i) = s_k(l) \) and \( l = 1 \) (\( l \) is even). In order to simplify (23), define the following \( N + 1 \) random variables:

\[
F_j = \begin{cases} 
    b_i(-1)s_k(N + 2j - l + 1)s_k(2j + 1), & j = 0, 1, \ldots, \frac{N}{2} - 1 \\
    b_i(0)s_k(2j - l + 1)s_k(2j + 1), & j = \frac{N}{2}, \ldots, N - 2 \\
    b_i(0)s_k(N - l - 1)s_k(N - 1), & j = \frac{N}{2} - 1 \\
    b_i(-1)s_k(N - l - 1)s_k(0), & j = \frac{N}{2} 
\end{cases}
\]

(24)

and

\[
G_j = \begin{cases} 
    b_i(-1)s_k(N + 2j - l + 1)s_k(2j + 1), & j = 0, 1, \ldots, \frac{N}{2} - 1 \\
    b_i(0)s_k(2j - l + 1)s_k(2j + 1), & j = \frac{N}{2}, \ldots, N - 2 \\
    b_i(0)s_k(N - l - 1)s_k(N - 1), & j = \frac{N}{2} - 1 \\
    b_i(-1)s_k(N - l - 1)s_k(0), & j = \frac{N}{2} 
\end{cases}
\]

(25)

For any \( l \) in the set \{0, 2, ..., N/2−2\}, the random variables \( F_j \) and \( G_j \) are mutually independent and satisfy:

\[
\Pr(F_j = +1) = \Pr(F_j = -1) = \frac{1}{2}
\]

(26)

Using the definitions of the random variables \( F_j \) and \( G_j \) and the fact that \( s_k(l) = 1 \) for every \( j \), (23) can be simplified to:

\[
W_i^e = \sum_{j=0}^{N/2 - 1} F_j \left( h_{1/2}(r) + s_k(2j + 1) \right)
\]

\[
\times s_k(2j + 2)h_{1/2}(r) + F_{N/2 - 1} h_{1/2}(r) + F_N h_{1/2}(r)
\]

\[
+ \sum_{j=0}^{N/2 - 1} G_j \left( h_{1/2}(r) + s_k(2j + 1)h_{1/2}(r) \right)
\]

(27)

Define the set \( A \) to be the set of all non-negative integers less than \( N/2-1 \) such that \( s_k(2j+1)s_k(2j+2) = +1 \) and \( B \) to be the set of all non-negative integers less than \( N/2-1 \) such that \( s_k(2j+1)s_k(2j+2) = -1 \). Similarly, define the set \( C \) to be the set of all nonnegative integers less than \( N/2 \) such that \( s_k(2j+1)s_k(2j+2) = +1 \). It follows from the definitions of the sets \( A, B, C \) and \( D \) that (27) can be written as follows:

\[
W_i^e = \sum_{j \in A} F_j \left( h_{1/2}(r) + h_{1/2}(r) \right)
\]

\[
+ \sum_{j \in B} F_j \left( h_{1/2}(r) - h_{1/2}(r) \right)
\]

\[
+ F_{N/2 - 1} h_{1/2}(r) + F_N h_{1/2}(r)
\]

\[
+ \sum_{j \in C} G_j \left( h_{1/2}(r) + h_{1/2}(r) \right)
\]

\[
+ \sum_{j \in D} G_j \left( h_{1/2}(r) - h_{1/2}(r) \right)
\]

(28)

Now restore the index \( j \) and define \( X_j = \sum_{i \in A} F_i \), \( Y_i = \sum_{i \in B} F_i \), \( P_i = F_{N/2 - 1} \), \( Q_i = F_N \) and \( U_i = \sum_{i \in C} G_i \), \( V_i = \sum_{i \in D} G_i \).
Then:

\[ W_i^* = X_i [h_2(r_i) + h_1(r_i)] + Y_i [h_2(r_i) - h_1(r_i)] + P_i h_2(r_i) + Q_i h_1(r_i) + U_i [h_2(r_i) + h_1(r_i)] + V_i [h_2(r_i) - h_1(r_i)] \]  

(29)

Similarly, if \( i \) is odd, then it can be shown that \( W_i \) in (22) is given by:

\[ W_i = X_i [h_2(r_i) + h_1(r_i)] + Y_i [h_2(r_i) - h_1(r_i)] + P_i h_2(r_i) + Q_i h_1(r_i) + U_i [h_2(r_i) + h_1(r_i)] + V_i [h_2(r_i) - h_1(r_i)] \]  

(30)

From the definitions of the random variables \( X_i, Y_i, P_i, Q_i, U_i \) and \( Y_i \), it is not hard to see that these random variables are mutually independent given \(|B|\) and \(|D|\) (which, respectively, are the cardinalities of sets \( B \) and \( D \)). Furthermore, the random variables \( W_i^* \) and \( W_i \), \( i = 1, \ldots, K \) are also mutually independent. This follows from the fact that these random variables are functions of elements in disjoint subsets of mutually independent random variables [2]. The random variables \( P_i \) and \( Q_i \) are uniformly distributed over \( \{0,1\} \). Given \(|B|\) and \(|D|\), the density functions of \( X_i, Y_i, U_i \) and \( V_i \) can be determined by elementary combinatorial arguments [2] but they are not needed in deriving the improved GA for double chip waveforms. Only the first and second moments of these random variables are important and they are given by \( E(X) = E(Y) = E(U) = E(V) = 0 \), \( E(X^2) = |A| = N - 2 - |B| - 1 \), \( E(Y^2) = |B| \), \( E(U^2) = |C| = N - 2 - |D| \), \( E(V^2) = |D| \). Furthermore, the first and second moments of \(|B|\) and \(|D|\) can be shown to be \( E(|B|) = (N - 2)/4 \), \( E(|D|^2) = N(N - 2)/16 \), \( E(|D|) = N/4 \) and \( E(|D|^2) = N(N + 2)/16 \).

The second term of (4) is the multiple access interference (MAI). The most straightforward approximation to the error probabilities is the standard GA, where the MAI is approximated by a Gaussian random variable. Using a Gaussian approximation, the error probability is given by \( p_i^G = Q(\sqrt{SNR}) \), where \( SNR \) is the signal-to-interference plus noise ratio at the output of the correlation receiver. The standard GA is clearly very simple but is not accurate in general. It is very optimistic when the signal to noise ratio increases.

Let \( r = [r_1, r_2, \ldots, r_K], \varphi = [\varphi_1, \varphi_2, \ldots, \varphi_K] \) and \( \Psi = \varphi \) (MAI\( r, \varphi \) | \( |B| \), |\( D|\)). Since \( \Psi \) is a function of \( r, \varphi \) and |\( B|\), |\( D|\), \( \Psi \) can be thought of as a random variable. Let \( \mu \) and \( \sigma \) be the mean and standard deviation of \( \Psi \), then the Holtzman’s improved GA for the error probabilities is as follows [13]

\[ p_i^H = \frac{2}{3} Q\left[\left(\frac{\gamma_i}{\mu} + 2\mu/(PT^2)\right)^{-1/2}\right] + \frac{1}{6} Q\left[\left(\frac{\gamma_i}{\mu} + 2(\mu + \sqrt{3}\sigma)/(PT^2)\right)^{-1/2}\right] + \frac{1}{6} Q\left[\left(\frac{\gamma_i}{\mu} + 2(\mu - \sqrt{3}\sigma)/(PT^2)\right)^{-1/2}\right] \]  

(31)

where \( E_{th} = PT \) is the energy per symbol and \( \gamma_i = 2E_{th}/N_0 \). To use (31) one needs to determine \( \mu \) and \( \sigma \).

The random variable \( \Psi \) can be written as \( \Psi = \sum_{i=1}^K L_i \), where

\[ L_i = \frac{P}{4} [1 + \cos(2\varphi_i)] \text{var}(W_i | r_i, |B|, |D|) \]  

(32)

Note that \( L_i \) are identically distributed and conditionally independent, given \(|B|\) and \(|D|\). Let \( \alpha_i = \text{var}(W_i^* | r_i, |B|, |D|) \) and \( \beta_i = \text{var}(W_i^* | r_i, |B|, |D|) \). Then it can be shown that:

\[ \gamma_i = \frac{N}{2} \left[ \frac{N}{2} h_2^2(r_i) + N h_1^2(r_i) \right] + 2 \left( \frac{N}{2} - 1 - 2|B| \right) h_2(r_i) h_1(r_i) + \frac{N}{2} \left[ h_1^2(r_i) + h_2^1(r_i) \right] + 2 \left( \frac{N}{2} - 2|D| \right) h_1(r_i) h_2(r_i) \]  

(33)

\[ \beta_i = \frac{N}{2} \left[ h_2^2(r_i) + h_1^2(r_i) \right] + 2 \left( \frac{N}{2} - 1 - 2|B| \right) h_2(r_i) h_1(r_i) + \frac{N}{2} \left[ h_1^2(r_i) + h_2^1(r_i) \right] + 2 \left( \frac{N}{2} - 2|D| \right) h_1(r_i) h_2(r_i) \]  

(34)

Since \( l_i \) takes even or odd integers of the set \( \{0,1, \ldots, N-1\} \) with the same probability, \( \text{var}(W_i | r_i, |B|, |D|) \) equals \( \alpha_i \) or \( \beta_i \), with probability 1/2.

Now the mean of \( \Psi \) can be obtained by averaging over the random variables \( \varphi, r_i, |B| \), \(|D|\). It is given by:

\[ \mu = \sum_{i=1}^K E(L_i) = (K - 1) E(L_i) \]

\[ = \frac{(K - 1) PT^2}{16N} \int_0^{T_c} \left[ h_2^i(r) + \frac{h_2^2(r)}{2} \right] dr \]

\[ + \frac{h_1^2(r) + h_2^i(r)}{2} + h_1^2(r) + h_2^i(r) + h_1^2(r) + h_2^i(r) \]  

(35)

where the last equality follows from the following identity

\[ \int_0^{T_c} h_k^i(r) h_k^j(r) dr = \int_0^{T_c} h_k^i(r) h_k^j(r) dr \]  

(36)

where \( m, n, p, q \in \{1,2\} \) and \( k, l \) are any (positive) integer numbers. Furthermore, based on (8) and (36) it is easy to see that the integral in (35) is equal to \( I \times T_c^2 \). Thus \( \mu \) is given by:

\[ \mu = \frac{(K - 1) PT^2}{8N} I \]  

(37)

The variance of \( \Psi \) is calculated as follows:

\[ \sigma^2 = E(\Psi^2) - \mu^2 \]

\[ = (K - 1) E(L_i^2) \]

\[ + (K - 1)(K - 2) E(L_i L_j) - \mu^2 \]  

(38)

where \( i \) and \( j \) are any indices not equal to \( k \) and not equal to each other. It can be shown that the second moment of \( L_i \) is
given by
\[ E(L_i^2) = \frac{3P^2}{64} [E(x_i^2) + E(\beta_i^2)] \]
\[ = \frac{3(P^2)^2}{128N^2} w_N \]  \hspace{1cm} (39)
where
\[ w_N = \frac{1}{T_c} \int_0^{T_c} \left[ \left( h_i^2(r) + h_j^2(r) \right)^2 + 4h_i^2(r)h_j^2(r) \right] \right) dr \]  \hspace{1cm} (40)

The correlation between \( L_i \) and \( L_j \) \((i\neq j)\) can be shown to be
\[ E(L_iL_j) = \frac{P^2}{64} [E(x_i x_j) + E(x_i, \beta_j) + E(\beta_i, \beta_j)] \]
\[ + E(\beta_i, \beta_j)] = \frac{(P^2)^2}{256N^2} \bar{w}_N \]  \hspace{1cm} (41)
where
\[ \bar{w}_N = \frac{1}{T_c} \int_0^{T_c} \left[ \left( h_i^2(r) + h_j^2(r) \right)^2 \right. \]
\[ + \frac{1}{T_c} \int_0^{T_c} \left( h_i^2(r) + h_j^2(r) \right) \]  \hspace{1cm} (42)

Finally, combining (37), (39) and (41), one has:
\[ \sigma = \frac{P^2}{16N} (K-1)^{1/2} \]
\[ \times \left[ 6w_N + (K-2)\bar{w}_N - 4(K-1)\bar{w}_N \right]^{1/2} \]  \hspace{1cm} (43)

Though the expressions for \( \mu \) and \( \sigma \) given above appear to be complicated, they are quite simple to evaluate. Also note that, although there are eight possible correlation functions that can be defined for the two chip waveforms, only four of them are needed in the evaluation of \( \mu \) and \( \sigma \). Furthermore, if the chip waveforms \( q_i(t) \) and \( q_j(t) \) possess an even or odd symmetry about \( T_c/2 \), then only three correlation functions are required and the expressions for \( w_N \) and \( \bar{w}_N \) can be significantly simplified [12]. Finally, the accuracy of the improved GA is verified in Section 5 against the simulation results.

5 Numerical examples

This Section evaluates the accuracy of the improved GA for error probabilities of DS-CDMA systems employing double chip waveforms. Through numerous examples, it also demonstrates the usefulness of using double chip waveforms over the single chip waveform, both in terms of SINR and BER.

Consider the following chip waveforms:
(i) Rectangular pulse: \( p_1(t) = p_{T_c}(t) \), where \( p_{T_c}(t) = 1 \) for \( 0 \leq t \leq T_c \) and \( p_{T_c}(t) = 0 \) otherwise.
(ii) Half-sine:
\[ p_2(t) = \sqrt{2} \sin \left( \frac{\pi t}{T_c} \right) p_{T_c}(t) \]
(iii) Raised cosine:
\[ p_3(t) = \sqrt{\frac{2}{3}} \left[ 1 - \cos \left( \frac{2\pi t}{T_c} \right) \right] p_{T_c}(t) \]
(iv) Blackman [10]:
\[ p_4(t) = c_1 \left[ 0.42 - 0.5 \cos \left( \frac{2\pi t}{T_c} \right) \right] + 0.08 \cos \left( \frac{4\pi t}{T_c} \right) \]  \hspace{1cm} (44)

where the constant \( c_1 \) is such that \( \int_0^{T_c} p_4(t) dt = T_c \).
(v) Four-term odd cosine series [22]:
\[ p_5(t) = \left[ 0.868 - 0.686 \cos \left( \frac{2\pi t}{T_c} \right) \right. \]
\[ + 0.149 \cos \left( \frac{4\pi t}{T_c} \right) - 0.033 \cos \left( \frac{6\pi t}{T_c} \right) \]  \hspace{1cm} (45)

where \( \int_0^{T_c} p_5(t) dt = T_c \).
(vi) Half-cosine:
\[ p_b(t) = \sqrt{2} \cos\left(\frac{\pi t}{T_c}\right) p_E(t) \]

(vii) Full-sine:
\[ p(t) = \sqrt{2} \sin\left(\frac{2\pi t}{T_c}\right) p_E(t) \]

Note that the first five chips are even about \( T_c/2 \), while the last two are odd.

The combinations of two chip waveforms used for the evaluation of the improved GA are ‘rectangular/half-sine’, ‘rectangular/raised-cosine’ and ‘rectangular/Blackman’. Since the exact calculation of the error probability for the case of double chip waveforms is not available, the results produced by the improved GA are compared with simulation results. Also compared are the results calculated based on standard Gaussian approximation. The error probabilities obtained by different methods are listed in Tables 1 and 2 for two systems with \( K = 3 \); \( N = 32 \) and \( K = 6 \); \( N = 64 \), respectively. From Tables 1 and 2, it can be seen that the accuracy of the improved GA is acceptable for all the signal-to-noise ratios, whereas the standard GA is quite loose for \( E_b/N_0 \gtrsim 8 \text{dB} \).

As mentioned before, while optimal chip waveform design is not the topic of this paper, here we shall demonstrate the usefulness of the proposed technique based on the popular chip waveform shapes listed above.

The values of interference parameter \( I \) are given in Table 3 for all combinations of the above chip waveforms. As expected, using the even chips in combination with the odd chips reduces the interference significantly (see columns 6 and 7 of Table 3). It is also of interest to note that using two odd chips offers lower interference compared to using two even chips. It should be noted, however, that the PSDs of the odd chips usually spread wider than that of the even chips and the increase in bandwidth may offset the interference reduction. The values of time-bandwidth product WTc for some commonly used values of \( \eta \) (1% and 10%) are tabulated in Table 4. As discussed before, for a fair comparison the parameter \( IWT_c \) needs to be considered and values are given in Table 5.

From Table 5 one can see that, even when the effect of increasing bandwidth of the odd chip is taken into account, it is still beneficial to combine the full-sine chip (odd chip) with the even chips for both FOBP bandwidth criteria. Another observation is that, although the half-cosine is an odd chip, it has very large FOBP bandwidth (due to its discontinuities). Hence there is no advantage to combine this odd chip with other even chip waveforms.

Finally, Figs. 1 and 2 show the advantage of using double chip waveforms over a single chip waveform in terms of the BER for systems with \( \eta = 10\% \) and \( \eta = 1\% \), respectively. The error probabilities are calculated based on (31). Note that in both cases, the FOBP transmission bandwidth is selected so that the processing gain of the corresponding system using a single raised cosine waveform equals \( N = 64 \) (i.e. \( WT_c = 64 \times 0.9501 \) for \( \eta = 10\% \) and \( WT_c = 64 \times 1.4093 \) for \( \eta = 1\% \)). The number of users in both systems is \( K = 8 \).

Clearly, Figs. 1 and 2 consistently verify the advantage of using double chip waveforms \( p_b(t)/p_E(t) \) (raised cosine/full-sine) over other single chip waveforms in terms of the BER performance.

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### Table 1: Error probabilities of DS-CDMA systems with double chip waveforms \((K=3, N=32)\)

<table>
<thead>
<tr>
<th>( E_b/N_0 )</th>
<th>Rect./half-sine ( p^G_e )</th>
<th>( p^H_e )</th>
<th>( P_e )</th>
<th>Rect./raised cosine ( p^G_e )</th>
<th>( p^H_e )</th>
<th>( P_e )</th>
<th>Rect./Blackman ( p^G_e )</th>
<th>( p^H_e )</th>
<th>( P_e )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>4.196</td>
<td>4.196</td>
<td>4.226 ( \times 10^{-2} )</td>
<td>4.149</td>
<td>4.149</td>
<td>4.152 ( \times 10^{-2} )</td>
<td>4.117</td>
<td>4.117</td>
<td>4.124 ( \times 10^{-2} )</td>
</tr>
<tr>
<td>4</td>
<td>1.617</td>
<td>1.622</td>
<td>1.644 ( \times 10^{-2} )</td>
<td>1.577</td>
<td>1.581</td>
<td>1.611 ( \times 10^{-2} )</td>
<td>1.550</td>
<td>1.553</td>
<td>1.593 ( \times 10^{-2} )</td>
</tr>
<tr>
<td>6</td>
<td>4.305</td>
<td>4.397</td>
<td>4.399 ( \times 10^{-2} )</td>
<td>4.072</td>
<td>4.152</td>
<td>4.205 ( \times 10^{-2} )</td>
<td>3.920</td>
<td>3.992</td>
<td>3.983 ( \times 10^{-2} )</td>
</tr>
<tr>
<td>8</td>
<td>7.220</td>
<td>8.079</td>
<td>8.113 ( \times 10^{-2} )</td>
<td>6.420</td>
<td>7.143</td>
<td>7.256 ( \times 10^{-2} )</td>
<td>5.919</td>
<td>6.557</td>
<td>6.516 ( \times 10^{-2} )</td>
</tr>
<tr>
<td>10</td>
<td>0.727</td>
<td>1.125</td>
<td>1.159 ( \times 10^{-4} )</td>
<td>5.746</td>
<td>8.848</td>
<td>9.122 ( \times 10^{-4} )</td>
<td>4.874</td>
<td>7.457</td>
<td>7.832 ( \times 10^{-4} )</td>
</tr>
<tr>
<td>12</td>
<td>0.460</td>
<td>1.581</td>
<td>1.620 ( \times 10^{-6} )</td>
<td>0.298</td>
<td>1.062</td>
<td>1.154 ( \times 10^{-6} )</td>
<td>2.184</td>
<td>7.930</td>
<td>9.091 ( \times 10^{-6} )</td>
</tr>
<tr>
<td>14</td>
<td>0.220</td>
<td>2.939</td>
<td>3.213 ( \times 10^{-6} )</td>
<td>0.106</td>
<td>1.661</td>
<td>1.712 ( \times 10^{-6} )</td>
<td>0.062</td>
<td>1.084</td>
<td>1.231 ( \times 10^{-6} )</td>
</tr>
</tbody>
</table>

\( p^G_e \): Gaussian approximation, \( p^H_e \): Holtzman’s approximation, \( P_e \): simulation

### Table 2: Error probabilities of DS-CDMA systems with double chip waveforms \((K=6, N=64)\)

<table>
<thead>
<tr>
<th>( E_b/N_0 )</th>
<th>Rect./half-sine ( p^G_e )</th>
<th>( p^H_e )</th>
<th>( P_e )</th>
<th>Rect./raised cosine ( p^G_e )</th>
<th>( p^H_e )</th>
<th>( P_e )</th>
<th>Rect./Blackman ( p^G_e )</th>
<th>( p^H_e )</th>
<th>( P_e )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>4.308</td>
<td>4.308</td>
<td>4.421 ( \times 10^{-2} )</td>
<td>4.248</td>
<td>4.248</td>
<td>4.290 ( \times 10^{-2} )</td>
<td>4.209</td>
<td>4.209</td>
<td>4.282 ( \times 10^{-2} )</td>
</tr>
<tr>
<td>4</td>
<td>1.714</td>
<td>1.716</td>
<td>1.815 ( \times 10^{-2} )</td>
<td>1.662</td>
<td>1.664</td>
<td>1.718 ( \times 10^{-2} )</td>
<td>1.628</td>
<td>1.630</td>
<td>1.648 ( \times 10^{-2} )</td>
</tr>
<tr>
<td>6</td>
<td>4.876</td>
<td>4.932</td>
<td>4.796 ( \times 10^{-3} )</td>
<td>4.567</td>
<td>4.616</td>
<td>4.662 ( \times 10^{-3} )</td>
<td>4.366</td>
<td>4.410</td>
<td>4.471 ( \times 10^{-3} )</td>
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<tr>
<td>10</td>
<td>1.197</td>
<td>1.503</td>
<td>1.476 ( \times 10^{-4} )</td>
<td>0.925</td>
<td>1.161</td>
<td>1.119 ( \times 10^{-4} )</td>
<td>7.699</td>
<td>9.662</td>
<td>9.491 ( \times 10^{-4} )</td>
</tr>
<tr>
<td>12</td>
<td>1.124</td>
<td>2.153</td>
<td>2.242 ( \times 10^{-5} )</td>
<td>0.712</td>
<td>1.407</td>
<td>1.424 ( \times 10^{-5} )</td>
<td>0.511</td>
<td>1.031</td>
<td>1.104 ( \times 10^{-5} )</td>
</tr>
<tr>
<td>14</td>
<td>0.938</td>
<td>3.811</td>
<td>4.356 ( \times 10^{-6} )</td>
<td>0.450</td>
<td>2.070</td>
<td>2.432 ( \times 10^{-6} )</td>
<td>0.262</td>
<td>1.314</td>
<td>1.223 ( \times 10^{-6} )</td>
</tr>
</tbody>
</table>

\( p^G_e \): Gaussian approximation, \( p^H_e \): Holtzman’s approximation, \( P_e \): simulation
Table 3: Values of $I$ for all combinations of chip waveforms

<table>
<thead>
<tr>
<th>$p_1(t)$</th>
<th>$p_2(t)$</th>
<th>$p_3(t)$</th>
<th>$p_4(t)$</th>
<th>$p_5(t)$</th>
<th>$p_6(t)$</th>
<th>$p_7(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.3333</td>
<td>1.2346</td>
<td>1.1028</td>
<td>1.0147</td>
<td>1.1840</td>
<td>0.7280</td>
<td>0.6836</td>
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<tr>
<td>1.1733</td>
<td>1.0592</td>
<td>0.9786</td>
<td>1.1303</td>
<td>0.7680</td>
<td>0.7168</td>
<td></td>
</tr>
<tr>
<td>0.9622</td>
<td>0.8920</td>
<td>1.0225</td>
<td>0.7327</td>
<td>0.6977</td>
<td></td>
<td></td>
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<tr>
<td>0.8291</td>
<td>0.9456</td>
<td>0.6983</td>
<td>0.6767</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.0887</td>
<td>0.7557</td>
<td>0.7088</td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>0.7680</td>
<td>0.7506</td>
<td></td>
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<tr>
<td>0.7933</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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</tbody>
</table>

$p_1(t)$ – rectangular, $p_2(t)$ – half-sine, $p_3(t)$ – raised cosine, $p_4(t)$ – Blackman, $p_5(t)$ – cosine series, $p_6(t)$ – half-cosine, $p_7(t)$ – full-sine

Table 4: Values of $WT_c$ for all combinations of chip waveforms and for different values of $\eta$

<table>
<thead>
<tr>
<th>$\eta$ = 10%</th>
<th>$p_1(t)$</th>
<th>$p_2(t)$</th>
<th>$p_3(t)$</th>
<th>$p_4(t)$</th>
<th>$p_5(t)$</th>
<th>$p_6(t)$</th>
<th>$p_7(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_1(t)$</td>
<td>0.8487</td>
<td>0.7886</td>
<td>0.9432</td>
<td>1.0987</td>
<td>0.8381</td>
<td>1.6249</td>
<td>1.3257</td>
</tr>
<tr>
<td>$p_2(t)$</td>
<td>0.7769</td>
<td>0.8665</td>
<td>0.9523</td>
<td>0.8070</td>
<td>1.1413</td>
<td>1.2026</td>
<td></td>
</tr>
<tr>
<td>$p_3(t)$</td>
<td>0.9501</td>
<td>1.0305</td>
<td>0.8946</td>
<td>1.2029</td>
<td>1.2304</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$p_4(t)$</td>
<td>1.1091</td>
<td>0.9784</td>
<td>1.3040</td>
<td>1.2750</td>
<td></td>
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<tr>
<td>$p_5(t)$</td>
<td>0.8366</td>
<td>1.1603</td>
<td>1.2112</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$p_6(t)$</td>
<td>2.0669</td>
<td>1.4671</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>$p_7(t)$</td>
<td></td>
<td>1.3564</td>
<td></td>
<td></td>
<td></td>
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<td></td>
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</table>

<table>
<thead>
<tr>
<th>$\eta$ = 1%</th>
<th>$p_1(t)$</th>
<th>$p_2(t)$</th>
<th>$p_3(t)$</th>
<th>$p_4(t)$</th>
<th>$p_5(t)$</th>
<th>$p_6(t)$</th>
<th>$p_7(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_1(t)$</td>
<td>10.2860</td>
<td>5.2471</td>
<td>5.2154</td>
<td>5.2154</td>
<td>4.8336</td>
<td>15.1610</td>
<td>5.3346</td>
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<tr>
<td>$p_2(t)$</td>
<td>1.1820</td>
<td>1.3490</td>
<td>1.5966</td>
<td>1.2472</td>
<td>10.0823</td>
<td>1.7272</td>
<td></td>
</tr>
<tr>
<td>$p_3(t)$</td>
<td>1.4093</td>
<td>1.5720</td>
<td>1.3811</td>
<td>10.0780</td>
<td>1.6834</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$p_4(t)$</td>
<td>1.6805</td>
<td>1.6265</td>
<td>10.0790</td>
<td>1.7660</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$p_5(t)$</td>
<td>1.3035</td>
<td>9.9859</td>
<td>1.7714</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$p_6(t)$</td>
<td>20.1487</td>
<td>9.1275</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$p_7(t)$</td>
<td>2.1971</td>
<td></td>
<td></td>
<td></td>
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<td></td>
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</tbody>
</table>

$p_1(t)$ – rectangular, $p_2(t)$ – half-sine, $p_3(t)$ – raised cosine, $p_4(t)$ – Blackman, $p_5(t)$ – cosine series, $p_6(t)$ – half-cosine, $p_7(t)$ – full-sine

6 SINR performance over a multipath Rayleigh fading channel

The analysis and the results obtained in the previous Sections are based on the AWGN channel model for DS-CDMA systems. Although the AWGN channel model is important when comparing different methodologies and it is also valid in many applications of spread spectrum (for example, in cable TV standards [23]), it is of interest to evaluate the effectiveness of the proposed technique for a multipath Rayleigh fading channel, a channel model typically seen in wireless mobile communications.

To this end, consider the following low-pass equivalent impulse response of the passband channel for the link between the $k$th user’s transmitter and a central station receiver [24]:

$$\lambda_k(t) = \sum_{j=1}^{L} |\zeta_{jk}|^2 \delta(t - \tau_{jk}) e^{j\phi_{jk}}$$  \hspace{1cm} (44)

In (44) $L$ is the number of resolved paths; $\zeta_{jk}$, $\tau_{jk}$ and $\phi_{jk}$ are the gain, delay and phase shift associated with the $k$th path, respectively. The maximum number of resolved paths is $L = \left\lceil T_{m}/T_c \right\rceil + 1$, where $T_{m}$ is the maximum multipath delay spread and $\left\lceil x \right\rceil$ returns the largest integer less than or equal to $x$. Both the delay and phase shifts are modelled as uniform random variables in $[0,T_c]$ and $[0,2\pi]$, respectively. The path gain $\zeta_{jk}$ is a Rayleigh random variable having the following probability density function (PDF)

$$f_{z}(x) = \begin{cases} \frac{2}{x} \exp\left(\frac{-x^2}{2}\right), & x \geq 0 \\ 0, & \text{otherwise} \end{cases} \hspace{1cm} (45)$$

where $\rho_0 = \frac{1}{2}E(\zeta^2)$. Also, as a slowly fading channel is assumed, the variables in (44) are assumed random but time-invariant.

With the $k$th user’s transmitted signal given in (1) and the channel model in (44), the received signal can be written as

$$y_{\text{fading}}(t) = \sum_{i=1}^{L} \sum_{k=1}^{K} \zeta_{jk} \sum_{i=-\infty}^{\infty} \sqrt{2P_h}(i) \times s_k(t - iT - \tau_{jk}) \times \cos(2\pi f_c(t - \tau_{jk}) + \phi_{jk}) + n(t) \hspace{1cm} (46)$$

where, similar to the case of AWGN channel, $n(t)$ is additive white Gaussian noise with two-sided power spectral
Table 5: Values of $\text{IWT}_c$ for all combinations of chip waveforms and for different values of $\eta$

<table>
<thead>
<tr>
<th>$\eta$ (dB)</th>
<th>$p_1(t)$</th>
<th>$p_2(t)$</th>
<th>$p_3(t)$</th>
<th>$p_4(t)$</th>
<th>$p_5(t)$</th>
<th>$p_6(t)$</th>
<th>$p_7(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10%$</td>
<td>1.1316</td>
<td>0.9735</td>
<td>1.0402</td>
<td>1.1149</td>
<td>0.9923</td>
<td>1.1829</td>
<td>0.9062</td>
</tr>
<tr>
<td>$1%$</td>
<td>13.7143</td>
<td>6.4781</td>
<td>5.7515</td>
<td>5.2921</td>
<td>5.7230</td>
<td>11.0372</td>
<td>3.6467</td>
</tr>
</tbody>
</table>

$p_1(t)$ – rectangular, $p_2(t)$ – half-sine, $p_3(t)$ – raised cosine, $p_4(t)$ – Blackman, $p_5(t)$ – cosine series, $p_6(t)$ – half-cosine, $p_7(t)$ – full-sine

Fig. 1 Comparison of error performance of asynchronous DS-CDMA systems using raised cosine (RC)/full sine (FS) double chip waveforms with 3 single chip waveforms: AWGN channel, $K=8$ with FOBP bandwidth of $W_T = 64 \times 0.9501$, $\eta = 10\%$

Fig. 2 Comparison of error performance of asynchronous DS-CDMA systems using raised cosine (RC)/full sine (FS) double chip waveforms with 3 single chip waveforms: AWGN channel, $K=8$ with FOBP bandwidth $W_T = 64 \times 1.4093$, $\eta = 1\%$

density of $N_0/2$; $\tau_{kl}$ and $\phi_{kl}$ are uniform random variables in $[0, T]$ and $[0, 2\pi]$, respectively.

As before, consider the detection of $b_i(0)$, i.e. the first information symbol of the $k$th user. Assume that the receiver can ideally lock on to the term at delay $\tau_{kl}$ and phase $\phi_{kl}$. Then the $j$th decision variable for detection is given by [24]

$$Z_{jk} = \int_0^T y(t) w(t) \cos(2\pi f_c t + \phi_{jk})dt$$

$$= \xi_{jk} \sqrt{P} \tilde{b}_i(0) T + \sqrt{P} \sum_{l=1}^{L} \xi_{lk}$$

$$\times \cos(\phi_{kl} - \phi_{jk}) [b_i(-1) R_k(\xi_{ik})]$$

$$+ b_i(0) \tilde{R}_k(\xi_{ik})]$$

$$+ \sqrt{P} \sum_{l=1}^{L} \sum_{k=1}^{K} \xi_{lk} \cos(\phi_{hl} - \phi_{jk})$$

$$\times [b_i(-1) R_k(\xi_{ih}) + b_i(0) \tilde{R}_k(\xi_{ih})] + n$$ (47)

where $\xi_{kl} = \tau_{kl} - \tau_{ij}$ and $n$ is a Gaussian random variable with zero mean and variance $N_0T/4$. The partial cross-correlation functions $R_k(\tau)$, $\tilde{R}_k(\tau)$, $R_k(\tau)$ and $\tilde{R}_k(\tau)$ in (47) are defined exactly the same as in Section 2.
Similar to $I_{k,l}$ in (5), define the following parameter:

$$I_{k,l} = [b_l(-1)R_{k,l}(\xi_{k,l}) + b_l(0)R_{k,l}(\xi_{k,l})] \cos (\phi_{k,l} - \phi_{l,k})$$  \hspace{1cm} (48)

With the above parameter, the decision variable in (47) can be rewritten as

$$Z_{jk} = \xi_{jk} \sqrt{\frac{P}{2}b_j(0)T \sum_{l=1}^{L} \xi_{jl}I_{k,l} + \frac{P}{2} \sum_{l=1}^{L} \sum_{l=1,j \neq k}^{K} \xi_{jl}I_{k,l} + n}$$  \hspace{1cm} (49)

The first term in (49) represents the desired signal to be detected. The second term is the self-interference of the desired user. The third term is the $L(K-1)$ multiple access interference (MAI) terms from the $K-1$ other simultaneous users of the system. Finally, the last term in (49) is due to additive white Gaussian noise.

Although the system’s error performance depends on what type of diversity reception is to be used, as far as the relative performance comparison is concerned, it suffices to compute the SINR associated with the detection variable $Z_{jk}$. For a fixed $\xi_{jk}$, it is easy to verify that the SINR is given by [24]

$$\mathcal{g} = \frac{E[Z_{jk}^2]}{\text{var}(Z_{jk})} = \frac{2x^2_Eb}{(L-1)E(\xi_{jk}^2)E_b \frac{1}{2\pi} + N_0}$$  \hspace{1cm} (50)

where the parameter $I$ is exactly the interference parameter of the double chip waveform defined in (17) and, as before, $E_b = PT$. Let $E_b = E(\xi_{jk}^2)E_b = E(\xi_{jk}^2)E_b = 2\rho_0E_b$ be the average received signal power, then the average value of SINR is

$$\frac{1}{2} = \left[1 + \sqrt{\frac{\rho_0}{\rho_0 + \frac{1}{2}}} \right]^L \times \frac{1}{\sum_{k=0}^{L-1} \left( \frac{L-1+k}{k} \right) \left[ \frac{1}{2} \left(1 + \sqrt{\frac{\rho_0}{\rho_0 + \frac{1}{2}}} \right)^k \right]}$$  \hspace{1cm} (51)

which is the same for every detection variable $Z_{jk}$ and is also very similar to the SINR obtained in (21) for AWGN channels.

The significance of the result provided by (51) is that maximising the average SINR is equivalent to minimising the normalised interference parameter $IWT_c$. This is exactly the same objective in selecting double chip waveforms for DS-CDMA systems over an AWGN channel considered in Section 3. Thus (51) clearly confirms the advantage of using double chip waveforms for multipath Rayleigh fading channels.

As mentioned earlier, the BER performance of DS-CDMA systems considered in this Section depends on the type of diversity reception used at the receiver [24]. Since neither exact calculation nor simple approximation to the BER is available, Gaussian approximation is still a popular technique for BER estimation of DS-CDMA systems over multipath Rayleigh fading channels [25–28]. If the diversity reception is maximum ratio combining (i.e. the RAKE receiver is used), then the gain and phase of each signal term of the desired user must be known. These gains and phases are then used to coherently combine individual path terms to form a single decision variable $\sum_{j=1}^{L} \xi_{jk}Z_{jk}$ for the $k$th user (assuming that the full-diversity order of $L$ is available). With this form of diversity reception, the BER based on Gaussian approximation is given by [24, 28]:

**Fig. 3** Comparison of the error performance of asynchronous DS-CDMA systems using the raised cosine (RC)/full sine (FS) double chip waveforms with 3 single chip waveforms: multipath Rayleigh fading channel, $K = 8$ with FOBP bandwidth $WT = 64 \times 1.4093$, $\eta = 1\%$, and $L = 4$
References


Appendix

Define the following correlation functions between the signature waveforms: $v_{k}(\tau) = \int_{-\infty}^{\infty} s_{k}(t) s_{k}(t + \tau) \, dt$ and $v_{k,i}(\tau) = \int_{-\infty}^{\infty} s_{k}(t) s_{i}(t + \tau) \, dt$. Note that with these definitions, $v_{k}(\tau) \neq v_{k,i}(\tau)$. Let $S_{k}(f) = F\{s_{k}(t)\}$ and $V_{k}(f) = F\{v_{k}(t)\}$, then $V_{k}(f) = |S_{k}(f)|^{2}$. Since $v_{k}(\tau)$ is an even function, time-limited to $[-T, T]$ and $R_{k}(\tau) = v_{k}(\tau)$ for $0 \leq \tau \leq T$, the right-hand side of (8) can be written as:

$$
2 \int_{0}^{T} R_{k}(\tau) R_{i}(\tau) \, d\tau = \int_{-\infty}^{\infty} v_{k}(\tau) v_{i}(\tau) \, d\tau
$$

$$
= \int_{-\infty}^{\infty} \left| S_{k}(f) \right|^{2} \left| S_{i}(f) \right|^{2} \, df
$$

(53)

$$
= \int_{0}^{T} \left[ R_{k}^{2}(\tau) + R_{i}^{2}(\tau) \right] \, d\tau
$$

(54)

Let $V_{k}(f) = F\{v_{k}(\tau)\}$. Write $v_{k,i}(\tau)$ as $v_{k,i}(\tau) = s_{k}(\tau) \otimes s_{i}(\tau)$ where $s_{i}(\tau) = s_{i}(\tau + T)$, then $V_{k,i}(f) = S_{k}(f) S_{i}^{*}(f) e^{-j2\pi f T}$. Define $f(\tau) = v_{k,i}^{*}(\tau) + v_{k}^{*}(\tau)$. Then $f(\tau)$ is time-limited to $[0, 2T]$. Furthermore, it can be shown that $f(\tau) = f(2T-\tau)$, i.e. $f(\tau)$ is an even function about $T$. Since $f(\tau) = R_{k}^{2}(\tau) + R_{i}^{2}(\tau)$ for $0 \leq \tau \leq T$, the right-hand side of (54) becomes:

$$
\int_{0}^{T} \left[ R_{k}^{2}(\tau) + R_{i}^{2}(\tau) \right] \, d\tau
$$

$$
= \frac{1}{2} \int_{0}^{2T} \left[ v_{k}^{*}(\tau) + v_{i}^{*}(\tau) \right] \, d\tau
$$

$$
= \frac{1}{2} \int_{-\infty}^{\infty} \left( |V_{k}(f)|^{2} + |V_{i}(f)|^{2} \right) \, df
$$

(55)

Finally, combining (53), (54) and (55) verifies (8).