SUMMARY Synchronous Gaussian code-division multiple access (CDMA) systems employing group-orthogonal signature waveforms are proposed and analyzed. All users in the system are divided into groups of users. The signature waveforms are constructed such that all the signature waveforms in one group are orthogonal to all the signature waveforms used in all other groups. This construction of signature waveforms ensures that there is no inter-group interference (i.e., among users in different groups), but at the expense of having intra-group interference (i.e., among users in the same group). However, by choosing a small size for each group, the intra-group interference can be effectively handled by a low-complexity, optimal (or suboptimal) multiuser detector. It is shown that a significant improvement in the system capacity can be achieved by the proposed technique over the conventional one that uses signature waveforms constructed from Welch-bound-equality (WBE) sequences. In particular, it is demonstrated that, while the conventional system’s error performance is very sensitive to even small amount of overload, the proposed system with an appropriate design of signature waveforms can achieve a much higher overload (up to 300% as shown in the paper) with an excellent error performance.

key words: signature waveforms, code-division multiaccess, WBE sequences, multiuser detection

1. Introduction

In a multiple access communication system, users transmit the information over the same channel at the same time and they are distinguished by their own signature waveforms. An important problem in the design of such a system is how to use limited resources such as power and bandwidth most efficiently to meet a given quality-of-service requirement, i.e., error performance. For a conventional synchronous CDMA system with $K$ users the major limitation to performance is multiple-access interference (MAI) caused by the correlation among the users’ signature waveforms, $s_k(t)$, $k = 1, 2, \ldots, K$. The amount of MAI depends on the number of users, $K$, relative to cardinality of the signal space, $N$ (usually called processing gain), used to construct the signature waveforms. If $K \leq N$ then an orthogonal set of waveforms can be chosen, eliminating the MAI completely and thereby achieving the performance of a single-user system.

The more interesting situation is when $K > N$ (i.e., the system is over-loaded or over-saturated) and one must inevitably encounter MAI. This is also a common situation recently considered in CDMA research literature. Different approaches have been developed to minimize the effect of MAI in this situation. In [1] signature waveforms with a tree-structured correlation property were proposed so that a low-complexity optimal multiuser detector (MUD) can be used. However it was not shown in [1] how to construct the signature set with the desired correlation property for arbitrary $K$ and $N$. A construction of signature waveforms in [2] does satisfy the tree-structured correlation property but it can only be applied to the case when $K = \frac{4N-1}{3}$. Recently the authors in [3]–[5] proposed a scheme where all users are divided into groups of orthogonal users and iterative interference cancellation technique is applied to minimize the MAI due to correlation among different groups.

Considerable attention has also been paid to the signature waveforms constructed from Welch-bound-equality (WBE) sequences [6]–[8]. The WBE signature waveforms are known to minimize the average MAI level [9] and they are the optimal signature waveforms for synchronous CDMA systems under various performance criteria [6]–[8]. However, the WBE waveforms are not necessarily optimal when the error performance of the users is considered**. With a specified bit-error-rate (BER) level, the MAI caused by WBE waveforms can still severely limit the number of users that can be supported in a synchronous CDMA system as the following discussion illustrates.

Consider a Gaussian multiple access system with processing gain (or dimensionality of the signal set) $N$. The actual value of $N$ is determined by the available bandwidth $W$ [10]. Let $K$ be the number of users that need to be accommodated in the system for a given bit-error-rate (BER) level. The case of interest is $K > N$, i.e., the system is overloaded. Assume that WBE sequences are used for minimum MAI. Then the signal-to-interference ratio at the receiver is $(K/N - 1)^{-1}$ [8]. The BER is given by*** $P_e \approx Q \left(\frac{K}{N - 1}\right)^{-1/2}$, where a Gaussian approxi-

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**It is surprising to see that the error performance of WBE waveforms was not investigated in the literature.

***$Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-t^2/2} dt$, $x \geq 0.$
mation is used for the MAI [11] and the background noise is ignored (since CDMA is interference-limited, not noise-limited). Define $K/N = \kappa > 1$, what can be considered to be an overload factor. Simple calculations show that for $P_e = 10^{-3}$, one must ensure $\kappa \leq 1.11$, i.e., there can be only up to $11\%$ overload in the system; at $P_e = 10^{-5}$, $\kappa \leq 1.06$, only $6\%$ overload is possible, etc. Clearly, the amount of allowable overload is very small.

As in all the previous work mentioned before, this paper also considers synchronous CDMA systems over additive white Gaussian noise (AWGN) channels. A novel technique is presented to increase the number of allowable users for a given system bandwidth (characterized by the number of orthogonal signal waveforms, or processing gain of the system). Here all users are divided into a number of small groups. The users’ signature waveforms are designed such that (i) signature waveforms in different groups are orthogonal (i.e., producing zero inter-group interference) and (ii) signature waveforms in the same group have a good correlation property so that performance degradation due to intra-group interference can be minimized. The key idea is that the size of each group is kept small so that the jointly optimal multiuser detector can be applied to effectively handle the intra-group interference. Note that the proposed technique is quite different from the one in [3]–[5] regarding the orthogonality property. In [3]–[5] the orthogonality holds among the users of the same group but not among different groups. Here, due to the group-orthogonality (orthogonality among different groups), no iterative interference cancellation among groups is required. This implies a much lower-complexity receiver for the proposed technique.

Although the additive white Gaussian noise CDMA channel model is the simplest channel model, such a model can be found in practical applications such as data-over-cable service [12]. Applying the technique proposed in this paper to other CDMA channel models (for example the fading channel models typically seen in mobile CDMA communications) is an interesting research problem.

The paper is organized as follows. Section 2 introduces the CDMA systems employing group-orthogonal signature waveforms. The design of the group-orthogonal signature waveforms to minimize the effect of the intra-group interference is the focus of Sect. 3. Section 4 presents the bit error rate performance and provides numerical results to demonstrate the superiority of the proposed CDMA system over the conventional one. Finally, conclusions are drawn in Sect. 5.

2. Proposed CDMA System

In a conventional CDMA system with $K$ users, the signature waveform $s_k(t)$ of the $k$th user is constructed as follows:

$$s_k(t) = \sum_{n=1}^{N} s_k(n) \phi_n(t), \quad k = 1, 2, \ldots K \quad (1)$$

where $\{\phi_1(t), \phi_2(t), \ldots, \phi_N(t)\}$ are some orthonormal basis functions, $N$ is called the processing gain and $\{s_k(1), s_k(2), \ldots, s_k(N)\}$ is the signature sequence of the $k$th user. Further the signature waveforms $s_k(t)$ are assumed to be unit-energy signals. The basis functions can be either time-limited or band-limited waveforms and their cardinality depends on the available bandwidth of the systems. For notational convenience, it is assumed in the rest of the paper that the basis functions are time-limited to the signaling interval. The received signal is given by

$$y(t) = \sum_{i=-\infty}^{\infty} \sum_{k=1}^{K} \sqrt{E_b} b_k(i) s_k(t - iT) + n(t) \quad (2)$$

where $T$ is the bit duration (the same for every user); $b_k(i)$ is the $i$th information bit of user $k$, taking values in $\{-1, +1\}$ with equal probability; $E_b$ is the energy per bit for each user and $n(t)$ is additive white Gaussian noise of spectral density $\sigma^2 = N_0/2$ (watts/Hz). As in any practical system, the transmission power is limited and therefore it is desired to minimize $E_b$ (or equivalently, to minimize the signal-to-noise ratio) to achieve a given system’s error performance. By projecting $y(t)$ onto each of the signature waveform, the sufficient statistic in the $i$th signaling interval can be obtained as

$$y_k(i) = \int_{iT}^{(i+1)T} y(t) s_k(t - iT) dt$$

$$= \sqrt{E_b} b_k(i) + \sqrt{E_b} \sum_{j=1, j \neq k}^{K} b_j(i) \rho_{jk} + n_k(i) \quad (3)$$

where $\rho_{jk} = \int_{0}^{T} s_j(t) s_k(t) dt$ is the correlation between the two signature waveforms $s_j(t)$ and $s_k(t)$, $n_k(i)$ is zero-mean Gaussian random variable with variance $N_0/2$. Note that, in (3) the first component is the desired signal component of user $k$, the second component is due to multiple access interference (MAI) from all other ($K - 1$) users and the last component is due to the background noise.

Different processing strategies can be implemented on the above sufficient statistic for the demodulation of the information symbols of the $K$ users (see [13], Sect. 2.9). It is straightforward to see that as long as the set of $K$ signature waveforms is not an orthonormal set, there exists MAI. If the matched filter is used, then the average variance (averaged over all users’ information bits and normalized by the power of the signal component) of the MAI is
\[
\text{var}\{\text{MAI}\} = \frac{1}{K} \sum_{k=1}^{K} \sum_{j=1, j \neq k}^{K} \rho_{jk}^2
\] (4)

It was shown in [8] that WBE sequences are optimal in terms of minimizing the above average variance of the MAI and the minimum MAI level is \((K/N - 1)\). It was also shown in [8] that if the minimum mean-square error (MMSE) receiver is employed, the WBE sequences are again optimal for minimizing the average variance of the MAI at the output of an MMSE receiver\(^1\). The limitation of MAI level on the BER performance of the conventional CDMA systems was discussed in Sect. 1. This limitation motivates a new proposal for CDMA system as shown in Figs. 1 and 2.

Consider the transmitter of the proposed CDMA system in Fig. 1. There are \(U\) groups, where each group consists of \(M = K/U\) users. The signal of the \(m\)th user in the \(u\)th group is spread by a distinct signature waveform \(s_{u,m}(t)\) before being simultaneously transmitted over the common CDMA channel. The users' signature waveforms are constructed as follows. From the set of \(N\) orthonormal basis functions \(\{\phi_1(t), \ldots, \phi_N(t)\}\), allocate to each group \(L = N/U = NM/K = M/\kappa\) distinct basis functions. With each set of these \(L\) basis functions we then construct \(M\) signature waveforms for \(M\) users in one group. For example, let \(\{\phi_{(u-1)L+l}(t)\}_{l=1}^{L}\) be the \(L\) orthonormal basis functions assigned to the \(u\)th group, then the \(M\) signature waveforms of \(M\) users in group \(u\) are

\[
s_{u,m}(t) = \sum_{l=1}^{L} s_{u,m}(l)\phi_{(u-1)L+l}(t)
\] (5)

where \(\{s_{u,m}(1), s_{u,m}(2), \ldots, s_{u,m}(L)\}\) is the signature of the \(m\)th user in the \(u\)th group.

\(^1\)In fact [8] shows that with WBE sequences, an MMSE receiver is identical to a MF receiver.
sequence of user \( m \) in group \( u \).

It is obvious that the above construction yields orthogonal signature waveforms across the groups. However, since \( L < M \) the signature waveforms of the same group are correlated, causing the intra-group interference. This observation suggests the receiver structure in Fig. 2 for the proposed transmission scheme. It consists of \( U \) multiuser detectors, each responsible for the signals of \( M \) users in the same group. Since the number of users in each group is typically kept small, the complexity of each MUD, even the jointly optimal MUD, in Fig. 2 is no longer an obstacle.

The received signal in the proposed scheme is given by

\[
y(t) = \sum_{i=-\infty}^{\infty} \sum_{p=1}^{U} \sum_{q=1}^{M} \sqrt{E_b} b_{p,q}(i) s_{p,q}(t-iT) + n(t),
\]

where \( b_{p,q}(i) \) is the information bit of user \( q \) in group \( p \) during the \( i \)th signaling interval. Without loss of generality, consider the detection of the information bits transmitted in the first bit interval. The output of the \( (u,m) \)th correlator which is matched to the signature waveform \( s_{u,m}(t) \) can be expressed as

\[
y_{u,m} = \int_{0}^{T} y(t) s_{u,m}(t) dt = \sqrt{E_b} b_{u,m}(0) + \sum_{i=1}^{M} \sqrt{E_b} b_{u,i}(0) \sum_{l=1}^{L} s_{u,m}(l) s_{u,i}(l) + n_{u,m},
\]

where \( n_{u,m} \) is a zero-mean Gaussian random variable with variance \( N_0/2 \). Equation (7) shows that \( y_{u,m} \) consists of three components, namely the desired signal component, the intra-group interference and a component due to the background noise. Since there is no interference from the other group, signal detection for users in group \( u \) can be implemented as a multiuser detection of a smaller CDMA system with \( M \) users in \( L \)-dimensional signal space.

To illustrate the above strategy, let’s consider a few examples. In the first example, there are \( K = 60 \) users in a CDMA system and the processing gain of the system is \( N = 40 \), i.e., the overload factor \( \kappa = 1.5 \). With the proposed technique, one can choose \( U = 20 \), \( M = 3 \) and \( L = 2 \), i.e., there are 20 orthogonal groups and each group consists of 3 users in a two-dimensional space. In the second example, \( K = 60 \) and \( N = 30 \), i.e., the overload factor \( \kappa = 2.0 \). Then one can design 15 orthogonal groups (\( U = 15 \)), each having 4 users (\( M = 4 \)) in a two-dimensional space (\( L = 2 \)). Note that in both cases, the values of \( M \) and \( L \) are very small.

The previous description of the proposed technique concentrates on the situation where all groups are allocated the same number of orthonormal basis functions and they have the same number of users. Of course the technique can be equally applied to the situation where different numbers of basis functions and users are assigned to each group. Thus in some situation, both overloaded and non-overloaded groups may exist in the system.

Obviously the performance of the proposed system depends on the signature waveforms used in each group, which in turn depends on the corresponding signature sequences. For the non-overloaded groups there is no need for the design of signature sequences since orthogonal sequences exist and they are the optimal sequences. The design of signature sequences for overloaded groups is considered in the next section. Note that if all groups have the same numbers of basis functions and users, they can have the same set of signature sequences. For simplicity, this is also the situation assumed in the rest of the paper.

3. Design of Signature Sequences for Each Group

Because the group index is not relevant in designing the signature sequences, it is dropped in this section for simplicity. Our problem is to design a set of \( M \) signature vectors (or sequences), each of length \( L \), i.e., \( s_m = [s_m(1), s_m(2), \ldots, s_m(L)]^\top, m = 1, 2, \ldots, M \) to optimize the error performance of the jointly optimal MUD.

Observe that with \( M \) signature sequences in \( L \) dimensional space, the “overload factor” for each user is \( M/L = \kappa \), exactly the same as that of the conventional system (i.e., a system with \( K \) users in \( N \) dimensional space). This implies that it is not helpful if WBE sequences are used in each group. Furthermore, it should be noted that WBE sequences minimize the total squared correlation [9] and they are only a good design for the matched filter or the MMSE receiver. With the proposed system, the jointly optimal multiuser receiver (see [13], Sect. 4.1) is implemented for each group, hence a better set of signature sequences needs to be found.

Unfortunately, the design of a set of \( M \) signature sequences that directly minimizes the bit error performance of the optimal MUD is difficult, if not impossible. This is due to the fact that no closed form expression for the bit error probability of the optimal MUD is available [13] (p.176). This is especially true when both \( M \) and \( L \) are large. Given this difficulty, an alternative criterion in our design of \( M \) signature sequences is to maximize the minimum distance among all the possible received signals of all users in the same group in the absence of the background noise (which shall be referred to as multiuser signals hereafter). This is a sensible criterion since the optimal MUD in the proposed receiver works with this constellation of multiuser signals and its asymptotic error performance is directly determined.
by the minimum distance among the multiuser signals. With this criterion, the design problem is developed as follows.

Let \( b = [b_1, b_2, \ldots, b_M]^T \) be the vector of the binary information bits of \( M \) users and \( y = [y_1, y_2, \ldots, y_M]^T \) be the vector of the sufficient statistic. Then it follows from (7) that \( y \) can be written as

\[
y = \sqrt{E_b} (S^T S) b + \sigma n = \sqrt{E_b} R b + \sigma n
\]

(8)

where \( S = [s_1, s_2, \ldots, s_M] \) is an \( L \times M \) signature matrix, \( R = S^T S \) is a \( M \times M \) correlation matrix of the signature waveforms with \( R_{mn} = \int_0^T s_m(t) s_n(t) dt = s_m^* s_n \), and \( n \) is a Gaussian vector of zero-mean with covariance matrix equal to \( R \) and independent of the transmitted bits.

Let \( \epsilon \) denote the normalized difference between any pair of distinct transmitted vectors of information bits of \( M \) users. The vector \( \epsilon \) is also called the error vector. The set of error vectors is (see [13], p.186)

\[
E = \{ \epsilon \in \{-1, 0, +1\}^M \}
\]

(9)

For example, let \( b_i \) and \( b_j \) be the two specific transmitted bit vectors (there are \( 2^M \) possible transmitted vectors of \( M \) users). Then the normalized error vector can be computed as \( \epsilon_{ij} = (b_i - b_j)/2 \). Using the error vectors, the normalized\(^1\) distance squared between the two multiuser signals that are associated with \( b_i \) and \( b_j \) can be computed as follows:

\[
\int_0^T \left( \sum_{m=1}^M b_i(m) s_m(t) - \sum_{m=1}^M b_j(m) s_m(t) \right)^2 dt = 4 (\epsilon_{ij}^T S \epsilon_{ij}) = 4 (\epsilon_{ij}^T R \epsilon_{ij})
\]

(10)

where \( b_i(m) \) extracts the information bit of user \( m \) in the transmitted bit vector \( b_i \). Although it appears from the above equation that all the distances can be determined from the set of error vectors and the correlation matrix \( R \) and that the design problem can be reduced to finding the correlation matrix \( R \). This is not true, however, due to the fact that it is not always possible to find an \( L \times M \) signature matrix \( S \) from a given \( M \times M \) correlation matrix \( R \).

The design problem can now be formulated in terms of the signature matrix \( S \) as follows.

**Problem 1**: Given two integers \( M \) and \( L \) with \( M > L \). Find an \( L \times M \) matrix \( S \) that maximizes

\[
d^2_{\text{min}} = \min_{\epsilon \in \{-1, 0, +1\}^M} \{ (\epsilon^T S^T S \epsilon) \}
\]

subject to the constraint that each column of \( S \) has unit norm.

As an example, consider solving Problem 1 for the simplest (but meaningful) case of \( L = 2 \) and \( M = 3 \). For \( L = 2 \), each signature sequence can be represented by a point on a unit circle. Let \( \theta_{1m} \) and \( \theta_{2m} \) be the angles between the signature sequence \( s_m \) and the two orthonormal basis functions \( \phi_1(t) \) and \( \phi_2(t) \) respectively. Obviously \( \theta_{2m} = \pi/2 - \theta_{1m} \) and therefore \( s_m = [\cos(\theta_{1m}), \cos(\theta_{2m})] = [\cos(\theta_{1m}), \sin(\theta_{1m})] \), \( m = 1, 2, 3 \). This means that the signature sequence \( s_m \) can be specified by one angle \( \theta_{1m} \), \( m = 1, \ldots, M \).

Since the objective function in (11) only depends on the relative locations of the signature sequences and does not depend on the sign of each signature sequence, one can assume without loss of generality that \( \theta_{11} = 0 \) and \( -\pi/2 \leq \theta_{1m} \leq \pi/2 \), \( m = 2, \ldots, M \).

Figure 3 illustrates the representation of three signature sequences on a two-dimensional signal space spanned by \( \phi_1(t) \) and \( \phi_2(t) \). For this specific case, it can be further assumed that \( 0 \leq \theta_{12} \leq \pi/2 \) and \( -\pi/2 \leq \theta_{13} \leq \pi/2 \). Furthermore, due to the symmetry involved, it can be expected that the optimal angles \( \theta_{12} \) and \( \theta_{13} \) that maximizes (11) are related by \( \theta_{13} = -\theta_{12} = -\theta \). This assumption is later verified by numerical results. With these assumptions, the eight multiuser signals with the corresponding bit patterns \( \{b_1b_2b_3\} \) are plotted in Fig.4. It is easy to verify that the two distances \( d_1 \) and \( d_2 \) determines the minimum distance. They are given by

\(^1\)The distance squared is normalized to \( E_b \).
From the above expressions, it is not hard to show that the minimum distance reaches a maximum value of \( d_{\text{min}} = 2\lfloor 2\cos(\pi/5) - 1 \rfloor \) when \( d_1 = d_2 \), or \( \theta = \pi/5 \). On the other hand, it is interesting to note that for \( L = 2 \) and \( M = 3 \), WBE sequences correspond to \( \theta = \pi/3 \) and they produce a minimum distance of zero among multiuser signals. The zero distance means that there is an error floor, regardless of what receiver is used. This observation clearly shows the inferiority of WBE sequences in terms of bit error rate performance.

It should also be pointed out that if the energy per bit \( E_b \) is the only constraint, then the constellation of eight multiuser signals in Fig.4 is not optimal in terms of maximizing the minimum distance. Many two-dimensional constellations, such as 8-phase shift keying (8-PSK) or 8-quadrature amplitude modulation (8-QAM), can produce a larger minimum distance for the same \( E_b \). It is, however, important to realize that there is a multiple access constraint on the design of multiuser signals for the CDMA systems as opposed to the design of an arbitrary multilevel/multiphase constellation. The multiple access constraint is further explained as follows. Recall that for the case of \( L = 2 \) and \( M = 3 \), each user in one group of the proposed CDMA systems is assigned a distinct signature sequence, represented by a vector in Fig.3. Without noise, the multiuser constellation in Fig.4 is formed by summing those vectors and their negative versions, where the sign of each vector is governed by the corresponding user’s information bit. Thus in essence the multiuser constellation in Fig.4 describes the received signal in the absence of the noise, not the transmitted signals. Clearly, it is impossible to design three vectors in Fig.3 to obtain an 8-PSK or 8-QAM constellation in Fig.4. In fact, our design yields the best 8-ary multiuser constellation for CDMA systems in terms of maximizing the minimum Euclidean distance.

Table 1 compares the minimum distance of the optimal multiuser constellation of the proposed CDMA system with that of various popular 8-ary constellations in Fig.5 [14]. Note that for each constellation in Fig.5 the minimum distance between the signal points is \( D_{\text{min}} = A \). Thus for a fair comparison, the minimum distance should be normalized to the average energy per bit. As a reference, the minimum distance corresponding to an orthogonal CDMA system is also shown. The table clearly shows the distance loss (i.e., performance loss) due to the multiple access constraint of the signal constellation. For example, compared to the best 8-point constellation in Fig.5(d) and the orthogonal system, the asymptotic losses of the optimum multiuser constellation in Fig.4 are about 20log_{10}(1.6330/1.2361) = 2.42 dB and 20log_{10}(2.0/1.2361) = 4.18 dB, respectively.

Unfortunately, a closed form solution to Problem 1 is not easy to obtained for arbitrary values of \( M \) and \( L \). However, Problem 1 can be solved numerically, for example by means of sequential quadratic programming (SQP) [15]. Note that with the small values of \( M \) and \( L \) as selected in this paper, the search for optimal signature sequences can be done easily. Table 2 presents the optimal signature sequences for different values of \( M \) and for \( L = 2 \) and \( L = 3 \). For \( L = 3 \), each sequence in Table 2 is represented by three angles in one column of matrix \( \theta = [\theta_{lm}] \), \( l = 1, \ldots, L \), \( m = 1, \ldots, M \). Since each signature sequence is represented by a point on a three-dimensional unit sphere, it is completely specified by two angles. This is because the three angles are related by \( \sum_{l=1}^{L} \cos^2(\theta_{lm}) = 1 \). The signature sequence for user \( m \) is \( s_m = [\cos(\theta_{1m}), \ldots, \cos(\theta_{Lm})]^T \).
4. Error Performance

The error performance of the proposed CDMA systems is determined by the performance of the jointly optimum MUD for $M$ users in each group. While closed-form expression for the error probability of the optimal MUD is not available, tight lower and upper bounds are known. In this section, the bounds in [13] shall be used. The justification of these bounds is not reproduced here and the interested reader is referred to [13].

Only the main results are summarized below.

Let $E_m = \{ \epsilon \in \{-1, 0, +1\}^M, \epsilon_m \neq 0 \}$ be the set of error vectors that affects the $m$th user. Note that $E = \bigcup_{m=1}^{M} E_m$. Let the number of nonzero components of an error vector and the energy of an hypothetical mutual signal modulated by $\epsilon$ are denoted, respectively, by $w(\epsilon) = \sum_{m=1}^{M} |\epsilon_m|$ and $\|Z(\epsilon)\|^2 = \int_0^T \left( \sum_{m=1}^{M} \epsilon_m s_m(t) \right)^2 dt = \epsilon^T \text{R} \epsilon$. An error vector $\epsilon \in E$ is decomposable into $\epsilon' \in E$ and $\epsilon'' \in E$ if [13] (p.187)

1. $\epsilon = \epsilon' + \epsilon''$;
2. if $\epsilon_m = 0$, then $\epsilon'_m = \epsilon''_m = 0$; and
3. $\epsilon'^T \text{R} \epsilon'' \geq 0$.

The subset of indecomposable vectors in $E_m$ is denoted by $F_m$. It can be shown that if $\epsilon \in F_m$ then $-\epsilon \in F_m$. Thus it is convenient to denote $F_+^m$ and $F_-^m$ the sets of indecomposable vectors in $F_m$ that have $\epsilon_m = 1$ and $\epsilon_m = -1$, respectively. Note that $F_m = F_+^m \cup F_-^m$. Let $d_{\text{min}} = \min \{ \|Z(\epsilon)\| \}$ and $w_{\text{min}} = \min_{\epsilon \in F_m, \|Z(\epsilon)\| = d_{\text{min}}} w(\epsilon)$.

With the above notations, the error probability of the optimum MUD for user $m$ is bounded as follows [13] (pp.188–194):

$$2^{-w_{\text{min}}} Q(\gamma d_{\text{min}}) \leq P_m[\text{error}] \leq \sum_{\epsilon \in F_+^m} 2^{-w(\epsilon)+1} Q(\|Z(\epsilon)\|)$$

(13)

where $\gamma = \frac{\sqrt{E_s}}{\sigma} = \sqrt{\frac{2P_s}{N_0}}$ is the signal to noise ratio. Note that, although one cannot design the signature sequences through the correlation matrix $\text{R}$, only the matrix $\text{R}$ is needed to evaluate the above lower and upper bounds. The correlation matrices for different sets of signature sequences found in Sect. 3 are listed in Table 3. Furthermore, finding the sets of indecomposable error vectors $F_m$, $m = 1, 2, \ldots, M$, for a given matrix $\text{R}$ is crucial in applying the above formulas.

Typically for an arbitrary signal correlation matrix, the indecomposable error vectors may only be found using an exhaustive search. If we find the indecomposable error vectors by checking the indecomposability of all error vectors, then the number of error vectors to be inspected is $3^M - 1$. However, using the property that for an indecomposable $\epsilon$ its negative version $-\epsilon$ is also indecomposable, we can reduce the search space by half. Thus in the exhaustive search, the number of error vectors to be inspected for indecomposability is $\frac{1}{2}(3^M - 1)$. This shows that, by keeping $M$ small, even the numerical evaluation of the error performance of the proposed technique is greatly simplified.

As an example, consider obtaining the bounds for the case $M = 3$ and $L = 2$. For this case, the optimum signature matrix is found previously to be

<table>
<thead>
<tr>
<th>Table 2</th>
<th>Optimum signature sequences (specified in terms of angles in degree).</th>
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<tbody>
<tr>
<td>$L = 2$</td>
<td>$M$</td>
</tr>
<tr>
<td>3</td>
<td>1.50</td>
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<tr>
<td>4</td>
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<tr>
<td>5</td>
<td>2.50</td>
</tr>
<tr>
<td>6</td>
<td>3.00</td>
</tr>
</tbody>
</table>

| $L = 3$ | $M$ | $\kappa = M/3$ | $\theta_{11}$ | $\theta_{12}$ | $\theta_{1M}$ | $\theta_{21}$ | $\theta_{22}$ | $\theta_{2M}$ | $\theta_{31}$ | $\theta_{32}$ | $\theta_{3M}$ | $d_{\text{min}}$ |
|---------|---------|--------------------------|
| 4 | 1.33 | 0 | 80.83 | -90 | 48.75 | 90 | 169 | 55.84 | $1.0508$ |
| 5 | 1.66 | 0 | 38.84 | 77.74 | 141.96 | 73.28 | $1.3035$ |
| 6 | 2.00 | 0 | 80.41 | 99.59 | 146.44 | 100.47 | 60.90 | $1.1547$ |
Table 3  Correlation matrix $R$ of the optimum signature sequences.

<table>
<thead>
<tr>
<th>$M$</th>
<th>$L = 2$</th>
<th>$L = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1.00</td>
<td>0.81</td>
</tr>
<tr>
<td></td>
<td>0.81</td>
<td>1.00</td>
</tr>
<tr>
<td></td>
<td>0.81</td>
<td>0.31</td>
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<td>0.87</td>
</tr>
<tr>
<td></td>
<td>0.87</td>
<td>0.00</td>
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</tbody>
</table>

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</thead>
<tbody>
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<td>0.25</td>
</tr>
<tr>
<td></td>
<td>0.25</td>
<td>1.00</td>
</tr>
<tr>
<td></td>
<td>0.94</td>
<td>-0.11</td>
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<tr>
<td></td>
<td>0.64</td>
<td>0.90</td>
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</table>

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<tbody>
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<td>0.43</td>
</tr>
<tr>
<td></td>
<td>0.43</td>
<td>1.00</td>
</tr>
<tr>
<td></td>
<td>0.81</td>
<td>-0.19</td>
</tr>
<tr>
<td></td>
<td>0.19</td>
<td>-0.89</td>
</tr>
<tr>
<td></td>
<td>0.12</td>
<td>-0.95</td>
</tr>
<tr>
<td></td>
<td>0.95</td>
<td>0.12</td>
</tr>
</tbody>
</table>

Table 4  Indecomposable error vectors ($M = 3$, $L = 2$).

<table>
<thead>
<tr>
<th>$\epsilon_1 = 1$</th>
<th>$\epsilon_1 = -1$</th>
<th>$\epsilon_2 = 1$</th>
<th>$\epsilon_2 = -1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_1$</td>
<td>$F_2$</td>
<td>$F_1$</td>
<td>$F_2$</td>
</tr>
<tr>
<td>$1 - 1 - 1$</td>
<td>$1 - 1 1$</td>
<td>$1 - 1 0$</td>
<td>$1 - 1 - 1$</td>
</tr>
<tr>
<td>$1 - 1 0$</td>
<td>$1 - 1 1$</td>
<td>$1 - 1 1$</td>
<td>$1 - 1 - 1$</td>
</tr>
<tr>
<td>$1 0 - 1$</td>
<td>$1 0 1$</td>
<td>$1 - 1 1$</td>
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<td>$1 0 0$</td>
<td>$1 0 0$</td>
<td>$1 0 - 1$</td>
<td>$1 0 - 1$</td>
</tr>
</tbody>
</table>

Table 5  Computations of $w(\epsilon)$, $Z(\epsilon)$ and $w_{m, \text{min}}$ ($M = 3$, $L = 2$).

| $\epsilon$ | $w(\epsilon)$ | $|Z(\epsilon)|$ | $w_{m, \text{min}}$ |
|------------|---------------|-----------------|------------------|
| $\setminus F_1^+$ | $\setminus F_2$ | $\setminus F_1^+$ | $\setminus F_2$ |
| $\epsilon_1 = 1$ | $1 - 1 - 1$ | $3$ | $2 \cos(\pi/5) - 1$ |
| $\epsilon_1 = 1$ | $1 - 1 0$ | $2$ | $2 \sin(\pi/10)$ |
| $\epsilon_1 = 1$ | $1 0 - 1$ | $2$ | $2 \sin(\pi/10)$ |
| $\epsilon_1 = 1$ | $1 0 0$ | $1$ | $1$ |

| $\epsilon_2 = 1$ | $-1 1 0$ | $2$ | $2 \sin(\pi/10)$ |
| $\epsilon_2 = 1$ | $-1 1 1$ | $3$ | $2 \cos(\pi/5) - 1$ |
| $\epsilon_2 = 1$ | $0 1 - 1$ | $2$ | $2 \sin(\pi/5)$ |
| $\epsilon_2 = 1$ | $0 1 0$ | $1$ | $1$ |

$$\mathbf{S} = \begin{bmatrix} \cos(\pi/5) & \cos(\pi/5) \\ \sin(\pi/5) & -\sin(\pi/5) \end{bmatrix}$$

and the correlation matrix is

$$\mathbf{R} = \begin{bmatrix} 1 & \cos(\pi/5) & \cos(\pi/5) \\ \cos(\pi/5) & 1 & \cos(2\pi/5) \\ \cos(\pi/5) & \cos(2\pi/5) & 1 \end{bmatrix}$$

Due to the symmetry of the signature sequences ($s_2$ and $s_3$ are symmetry with respect to $s_1$), $F_2 = F_3$ and users 2 and 3 have the same error performance. The sets of indecomposable vectors $F_1$ and $F_2$ are provided in Table 4, where each row (of 3 elements) is an indecomposable vector. Table 5 provides all the necessary calculations for the application of (13).

From Tables 5 and (13), the probability of error for each user is bounded as follows:

$$P_1^L = \frac{1}{2} Q\left(\frac{\gamma d_{\text{min}}}{2}\right) \leq P_1[\text{error}] \leq Q(\gamma)$$

$$+ Q(\gamma) + \frac{1}{4} Q\left(\frac{\gamma}{2}\right)$$

and

$$P_2^L = \frac{1}{2} Q\left(\frac{\gamma d_{\text{min}}}{2}\right) \leq P_2[\text{error}] \leq Q(\gamma)$$

$$+ \frac{1}{2} Q(\gamma) + \frac{1}{4} Q\left(\frac{\gamma}{2}\right)$$

Note that the average error probability, averaged over all three users, is bounded as

$$P[\text{error}] \leq \left(\frac{1}{3} P_1^L + \frac{2}{3} P_2^L\right) \leq \frac{\gamma d_{\text{min}}}{2} = \theta = 1.1756.$$

The bounds in (16) and (17) can also be expressed in terms of the minimum distance $d_{\text{min}}$ of the multiuser constellation in Fig. 4 as follows. Recall that for the case of $M = 3$ and $L = 2$, the maximum value of the minimum distance is achieved with $\theta = \pi/5$ and for this value of $\theta$, $d_1 = d_2 = d_{\text{min}} = 2[2 \cos(\pi/5) - 1] = \sqrt{5} - 1$. Moreover, since $2 \cos(\pi/5) - 1 = 2 \sin(\pi/10) = d_{\text{min}}/2 \approx 0.6180$ and $2 \sin(\pi/5) = 1.1756$, one has

$$\frac{1}{2} Q\left(\frac{\gamma d_{\text{min}}}{2}\right) \leq P_1[\text{error}] \leq Q(\gamma)$$

$$+ \frac{1}{2} Q\left(\frac{\gamma d_{\text{min}}}{2}\right) \approx 0.5 Q\left(\frac{\gamma d_{\text{min}}}{2}\right)$$

and

$$\frac{1}{2} Q\left(\frac{\gamma d_{\text{min}}}{2}\right) \leq P_2[\text{error}] \leq Q(\gamma) + \frac{1}{2} Q(1.1756\gamma)$$

$$+ \frac{3}{4} Q\left(\frac{\gamma d_{\text{min}}}{2}\right) \approx 0.75 Q\left(\frac{\gamma d_{\text{min}}}{2}\right)$$

The above expressions thus clearly confirm the role of the minimum distance on the error performance of the
proposed CDMA system.

Figure 6 plots the above lower and upper bounds. The lower bound is the same for all three users, while the upper bound of user 1 is worst than that of users 2 and 3. This is expected because with $\theta = \pi/5$ the signature sequence of user 1 is highly correlated to both the signature sequences of users 2 and 3 (see Fig. 4).

Although the lower and upper bounds in Fig. 6 are tight and should provide useful information about the actual error performance, it is of interest to investigate the error performance by means of computer simulation. To this end, the discrete model of the proposed CDMA systems in Eq. (7) is simulated. This means that the simulation results do not depend on the number of actual users $K$ and the particular choice of $N$ orthonormal basis functions. The results only depend on the size $M$ of each group and the number $L$ of orthonormal basis functions allocated for each group. In general, choosing which family of basis functions (such as Walsh functions, sinusoids) is governed by the bandwidth property of the family [10].

The error performances of users 1 and 2 obtained from computer simulation for the above system are plotted in Fig. 6. It was made sure in all simulations that at least 500 erroneous bits are observed for each user. It can be seen that the simulation results are within the corresponding lower and upper bounds and they follow the upper bounds quite closely. This suggests the tightness of the upper bound, which is consistent with the bounding technique developed in [13]. For comparison, the error performance of an orthogonal CDMA system is also plotted in Fig. 6. It can be observed that at the BER level of $10^{-4}$, the proposed technique achieves an overload factor of 1.5 at the expense of an additional 4.0 dB in signal-to-noise ratio. This observation is consistent with the loss predicted from the minimum distances in Table 1 before.

Note that if WBE sequences are used, i.e., if $\theta = \pi/3$ then the error probability of all three users exhibits an error floor of $1/8$. Recall that WBE sequences minimize the average MAI level which, similarly to (4), can be computed as $(\sum_{m=1}^{M} \sum_{n \neq m} R_{mn}^2) / M$. For the case of $M = 3$ and $L = 2$, the average MAI levels for WBE sequences and the designed sequences are 0.5 and 0.9363, respectively. Thus this example clearly shows that minimizing MAI is not necessary the same as minimizing the bit error rate of the jointly optimum multiuser detector.

To demonstrate the superiority of the proposed CDMA system over the conventional one (which uses WBE sequences), Figs. 7 and 8 plot the upper and lower bounds in (13) for different overload factors and when the signature sequences are designed in two-dimensional space ($L = 2$). The error performance of
the conventional CDMA system using WBE sequences and with an overload factor of only 1.5 is also shown in each figure. It is clear that, while the conventional system fails to deliver a satisfactory error performance even for such a small overload factor (regardless of how high a signal-to-noise ratio is), the proposed system achieves excellent error performance for much larger overloads. It can also be seen that the performance of the proposed system decreases as the overload factor increases. Furthermore, the lower and upper bounds become further away with a higher overload factor.

Compared to an orthogonal system (see Figs. 6 and 8), it is observed that an addition of 10 dB in signal-to-noise is required by the system with $\kappa = 3.0$ (i.e., an additional of 200% load) to achieve a BER level of $10^{-3}$. This result illustrates how power can be traded for the increased capacity by the proposed techniques, which is not possible by the conventional CDMA systems using WBE sequences as pointed out before. The result that about 10 dB excess power is required for the proposed system ($\kappa = 3.0$) to achieve the same BER as that of the orthogonal system ($\kappa = 1.0$) can be readily explained based on the minimum distance parameters of the two systems as listed in Tables 1 and 2: For a given transmitted power (i.e., given energy per bit), the normalized minimum distance of the orthogonal system is 2.0, which is about three times as that of the proposed CDMA system ($d_{\text{min}} = 0.6282$). Thus the difference is $20 \log_{10} (2.0/0.6282) = 10.06$ dB. It should also be pointed out that the above comparison is for single-cell CDMA systems. If multi-cell environment is considered then the frequency utilization factor needs to be taken into account in the comparison.

Figure 9 demonstrates how the error performance of the proposed system depends on the choice of group size. Here the overload factor $\kappa = 2.0$ is fixed. With this overload factor, the proposed system can be designed with $M = 4$, $L = 2$ (four users in a two-dimensional space), or $M = 6$, $L = 3$ (six users in a three-dimensional space). The figure shows that there is a gain to be achieved (about 1 dB in this case) by choosing a bigger group size. It should be noted, however, that this performance improvement comes at the expense of a higher complexity for MUD receiver. With the overload factor is fixed, a larger group size means a larger signal space. This makes it more flexible in designing the signature sequences. In fact the performance improvement with larger group size can be expected form Table 2 where it shows that the minimum distance achieved with $M = 6$, $L = 3$ is larger than that achieved by $M = 4$, $L = 2$.

It is acknowledged that the error performance of the scheme in [5] is better than the error performance of the technique proposed in this paper. However, the advantage of the proposed system is that it has a lower-complexity receiver where no iteration is required.

5. Conclusions

A novel technique of employing group-orthogonal signature waveforms (sequences) has been proposed to improve the user capacity of synchronous CDMA systems. The signature waveforms are designed to have zero inter-group interference and minimize the effect of intra-group interference. The residual intra-group interference can be effectively handled by a low-complexity optimal multiuser receiver. Performance analysis and numerical examples were provided to demonstrate the advantages of the proposed technique compared to the conventional technique that employs WBE sequences to minimize the multiple access interference across the whole system. In particular it was shown that, while the conventional CDMA systems can only accommodate a small amount of overload (up to 20%) at a very moderate error performance, the proposed systems can afford much higher overloads (up to 300% as demonstrated in the paper) with an excellent error performance.

Finally, although the technique proposed in this paper concentrates on the uncoded CDMA systems, its application to coded CDMA systems seems very attractive. This is especially true when iterative multiuser detection/decoding is applied as in [16]. In this case, due again to the small group size, the optimal soft-input soft-output (SISO) multiuser detection is feasible. Preliminary results showed that the performance of a single-user system (or orthogonal system) can be achieved after a few iterations. Detailed study of the proposed technique for coded CDMA systems will be reported in the future.

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References


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