

## **Welfare Reductions from Small Cost Reductions in Differentiated Oligopoly**

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X. Henry Wang and Jingang Zhao\*  
University of Missouri-Columbia  
University of Saskatchewan

**Abstract:** This paper compares the welfare effects of cost reductions in differentiated Bertrand and Cournot oligopolies. Using a linear demand system, the paper provides the critical levels of output shares below which a small cost reduction by a Bertrand or Cournot firm reduces welfare. It then shows that the critical value is larger under Bertrand competition than Cournot competition.

**Keywords:** Bertrand Equilibrium, Cournot Equilibrium, Cost reduction, Welfare effects

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\* Send correspondence to: Jingang Zhao, Department of Economics, University of Saskatchewan, 9 Campus Drive, Saskatoon, SK, Canada, S7N 5A5. Fax: (306) 966-5232, Tel: (306) 966-5217, Email: <j.zhao@usask.ca>. We would like to thank Professor Simon Anderson and two anonymous referees for valuable comments and suggestions.

## 1. Introduction

This paper studies the welfare effects of a single firm's cost reduction in Bertrand and Cournot oligopolies with differentiated goods. It continues the previous literature on the welfare effects of technological changes (or cost reductions) and shocks. Lahiri and Ono (1988) and Zhao (2001) provided conditions on market share for cost reductions to reduce welfare. Recently, Février and Linnemer (2004) provided conditions on variance or covariance to determine the welfare effects of shocks. Their analysis unified earlier studies on the effects of shocks by Bergstrom and Varian (1985), Salant and Shaffer (1999), Long and Soubeyran (2001), and Linnemer (2003). All of these papers focused on Cournot oligopolies with homogeneous goods.<sup>1</sup>

In contrast, the present paper derives analogous results for Bertrand and Cournot oligopolies with differentiated goods. It shows that in both models, a small cost reduction by one firm is welfare-decreasing if and only if its output share falls below a critical level determined by cost and demand parameters. It also shows that the critical value is larger under Bertrand competition than Cournot competition.

The rest of the paper is organized as follows. Sections 2 and 3 study the Bertrand model and the Cournot model with differentiated goods, respectively, and examine the effects of cost reductions in each model. Section 4 compares these effects; section 5 concludes the paper. The appendix provides proofs.

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<sup>1</sup> The exceptions are Anderson, de Palma, and Kreider (2001) and Bernstein and Federgruen (2004), which study respectively the profit effects of taxation and cost reduction in a Bertrand model.

## 2. The Bertrand Model

Consider the following Bertrand-Shubik demand system originated in Shubik (1980):

$$q_i(p) = q_i(p_1, \dots, p_n) = V - p_i - \gamma(p_i - \bar{p}), \quad (1)$$

where  $V > 0$  is the common quantity intercept,  $p_i$  is  $i$ 's price,  $\bar{p} = (\sum_{i=1}^n p_i)/n$  is the industry average price,  $\gamma \geq 0$  is the substitutability parameter, and the price space is given by  $P = \{p \in \mathbb{R}_+^n : V - p_i + \gamma(p_i - \bar{p}) > 0, i = 1, \dots, n\}$ . Firm  $i$ 's cost and profit functions are given by  $C_i(q_i) = c_i q_i$  and  $\pi_i(p) = (p_i - c_i)q_i(p)$ , where  $c_i > 0$  is its constant marginal (average) cost. For convenience, we label firms such that  $c_1 \leq c_2 \leq \dots \leq c_n$ , so firm 1 is the most efficient firm and firm  $n$  is the least efficient firm.

The Bertrand equilibrium is a price vector  $p^B = (p_1^B, \dots, p_n^B)$  such that each  $p_i^B$  is firm  $i$ 's best response to others' choices  $p_{-i}^B = (p_1^B, \dots, p_{i-1}^B, p_{i+1}^B, \dots, p_n^B)$ . Precisely, each  $p_i^B$  is the solution of  $\text{Max} \{ \pi_i(p_i, p_{-i}^B) \mid p_i \geq 0 \}$ , and  $p^B$  solves the following  $n$  first-order conditions:

$$\frac{\partial \pi_i(p)}{\partial p_i} = 0, \quad i = 1, \dots, n, \text{ or} \quad (2)$$

$$Bp = \begin{bmatrix} 2\delta & -\gamma & \cdots & -\gamma \\ -\gamma & 2\delta & \cdots & -\gamma \\ \vdots & \vdots & \ddots & \vdots \\ -\gamma & -\gamma & \cdots & 2\delta \end{bmatrix} \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix}, \text{ where} \quad (3)$$

$$\bar{c} = \frac{1}{n} \sum_{j=1}^n c_j, \quad \delta = n + (n-1)\gamma, \quad d_i = nV + \delta c_i \quad \text{for each } i. \quad (4)$$

Solving (3) for  $p$  leads to the following Bertrand equilibrium values:<sup>2</sup>

<sup>2</sup> The inverse matrices for solving (3) and (4), and for deriving (13) from (1), can all be obtained by applying Lemma 2 in Zhao and Howe (2004). Alternatively, as suggested by a referee, the inverse in (3) can be

$$p_i^B = \frac{n(V - \bar{c})}{2n + (n-1)\gamma} + \frac{n(1 + \gamma)\bar{c} + (n + (n-1)\gamma)c_i}{2n + (2n-1)\gamma}, \quad (5)$$

$$q_i^B = \frac{n + (n-1)\gamma}{n}(p_i^B - c_i), \quad Q^B = \sum q_i^B = \frac{n(n + (n-1)\gamma)}{2n + (n-1)\gamma}(V - \bar{c}), \text{ and} \quad (6)$$

$$\pi_i^B = \frac{n}{n + (n-1)\gamma}(q_i^B)^2. \quad (7)$$

We assume throughout that all firms produce positive outputs at the above Bertrand equilibrium, which is equivalent to the following assumption (A1):<sup>3</sup>

$$\text{A1: } \frac{nV(2n + (2n-1)\gamma) + \gamma(n + (n-1)\gamma)\sum_{j \neq n} c_j}{n^2(\gamma + 1)(\gamma + 2) - 2n\gamma(\gamma + 1) + \gamma^2} > c_n.$$

We will omit discussions about the average price, because  $\bar{p} = (\sum p_i)/n = V - \bar{Q} = V - (\sum q_i)/n$  follows directly from the industry output. Note that the markup  $(p_i^B - c_i)$  is given by

$$p_i^B - c_i = \frac{n(V - \bar{c})}{2n + (n-1)\gamma} + \frac{n(1 + \gamma)(\bar{c} - c_i)}{2n + (2n-1)\gamma},$$

so more efficient firms (i.e., firms with smaller  $c_i$ ) have larger markups and outputs. Note

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obtained by a known formula for  $B = \{I - \beta H\}/\alpha$ , where  $\alpha = 1/(2n + (2n-1)\gamma)$ ,  $\beta = \gamma\alpha < 1/n$ ,  $I$  is the identity matrix, and  $H$  is the matrix of ones. Specifically,  $B^{-1} = \alpha\{\sum_{k=0}^{\infty} [\beta H]^k\} = \alpha\{I + \sum_{k=0}^{\infty} [\beta^k n^{k-1} H]\} = \alpha\{I + [\beta/(1 - n\beta)]H\}$ . Multiplying  $B^{-1}$  into the right-hand side of (3) gives (5). Equations (6) and (7) follow directly by utilizing (5) and the demand equations in (1).

<sup>3</sup> For each  $k$  and  $(V, c_k) \in \mathbf{R}_{++}^n$ , A1 determines a feasible range  $(c_L, c_U)$  for our comparative statics analysis on  $c_k$ . It is not difficult to show that the upper bound  $c_U = c_U(k)$  and lower bound  $c_L = c_L(k)$  are

$$c_U = \frac{na_1V + \mathbf{g}a_2\sum_{j \neq k} c_j}{a_3}, \text{ and } c_L = \begin{cases} \frac{[a_3c_n - (na_1V + \mathbf{g}a_2\sum_{j \neq k, n} c_j)]^+}{\mathbf{g}a_2} & \text{if } k < n \\ \frac{[a_3c_{n-1} - (na_1V + \mathbf{g}a_2\sum_{j=1}^{n-2} c_j)]^+}{\mathbf{g}a_2} & \text{if } k = n, \end{cases}$$

where  $a_1 = 2n + (2n-1)\gamma$ ,  $a_2 = (n + (n-1)\gamma)$ , and  $a_3 = n^2(\gamma + 1)(\gamma + 2) - 2n\gamma(\gamma + 1) + \gamma^2$ , and  $t^+ = 0$  if  $t = 0$ ,  $t$  if  $t > 0$ .

that (6) implies the markup/output ratios are the same across all firms (i.e.,  $(p_i^B - c_i)/q_i^B = n/(n + (n - 1)\gamma)$  for all  $i$ ).

An inspection of (5)-(7) leads directly to the following effects of a small reduction in firm  $i$ 's marginal cost on prices, outputs and profits at the Bertrand equilibrium: (i) it decreases firm  $i$ 's price, and increases its output and profits; (ii) it decreases all other firms' prices, outputs and profits; and (iii) it increases the industry output.<sup>4</sup> Note that the own effect on firm  $i$ 's output dominates the combined output effects on all other firms, so that industry output increases after a cost reduction by firm  $i$ . Industry output increases because it depends on average cost (see (6)) and because  $i$ 's cost reduction reduces average cost.

The standard method used to compute consumer surplus is to assume a representative consumer with a quasi-linear utility function whose utility maximization, subject to a budget constraint, generates the demand system.<sup>5</sup> It is straightforward to verify that the utility function yielding the demand system in (1) is given by  $u(q) = a'q - q'Aq/2$ , where  $a' = (V, \dots, V)$  is an  $1 \times n$  vector,  $q$  denotes the  $n \times 1$  quantity vector, and

$$A = \frac{1}{1 + \mathbf{g}} \begin{bmatrix} 1 + \frac{\mathbf{g}}{n} & \frac{\mathbf{g}}{n} & \dots & \frac{\mathbf{g}}{n} \\ \frac{\mathbf{g}}{n} & 1 + \frac{\mathbf{g}}{n} & \dots & \frac{\mathbf{g}}{n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\mathbf{g}}{n} & \frac{\mathbf{g}}{n} & \dots & 1 + \frac{\mathbf{g}}{n} \end{bmatrix}. \quad (8)$$

The consumer surplus is given by  $CS = u(q) - p'q = q'Aq/2$ , which is obtained by applying the first-order conditions of utility maximization. Straightforward calculations show

<sup>4</sup> Closed form expressions of these effects are given in part 3 of Appendix B, available from the authors or at IJIO's supplementary material website (<http://www.mgmt.purdue.edu/centers/ijio/eo/eo.htm>).

<sup>5</sup> Spence (1976) provides an alternative formulation that yields the same measurement.

that, at the Bertrand equilibrium, industry profit  $\Pi^B = \sum_{j=1}^n \pi_j^B$ , total consumer surplus  $CS^B$ , and total welfare  $W^B$ , respectively, are given by

$$\Pi^B = \frac{n}{n + (n-1)\gamma} \sum_{j=1}^n (q_j^B)^2, \quad (9)$$

$$CS^B = \frac{1}{2(1+\gamma)} \sum_{j=1}^n (q_j^B)^2 + \frac{\gamma}{2n(1+\gamma)} (Q^B)^2, \text{ and} \quad (10)$$

$$W^B = \Pi^B + CS^B = \frac{3n + (3n-1)\gamma}{2(1+\gamma)(n + (n-1)\gamma)} \sum_{j=1}^n (q_j^B)^2 + \frac{\gamma}{2n(1+\gamma)} (Q^B)^2, \quad (11)$$

where  $q_j^B$  and  $Q^B$  are given by (6).

We now study how small changes in costs affect total profits, as well as consumer surplus and total welfare under price competition. Proposition 1 below characterizes such effects in terms of the magnitudes of marginal costs.

**Proposition 1.** *Under Bertrand competition, a small reduction in firm  $i$ 's marginal cost*

- (i) *reduces total profits  $\hat{U}$  if  $c_i$  is above a critical level  $c_i^{B*}$ ;*
- (ii) *always increases consumer surplus; and*
- (iii) *reduces total welfare  $\hat{U}$  if  $c_i$  is above a critical level  $c_i^{B**}$ , where the critical levels  $c_i^{B*}$  and  $c_i^{B**}$  are given in (24) and (26) in the appendix.*

For convenience, we call all firms with marginal costs below the industry average level (i.e.,  $\bar{c} - c_i > 0$ ) *efficient firms*, and all firms with marginal costs above the industry average level (i.e.,  $\bar{c} - c_i < 0$ ) *inefficient firms*. It is straightforward to show that  $c_i^{B**} > c_i^{B*}$ . Hence, if a cost reduction increases total profits, it also will increase total welfare. From the proof of Proposition 1, a small cost reduction by each efficient firm will raise total welfare. If cost reduction by an inefficient firm lowers total welfare, cost reduction by any less efficient firm also will lower total welfare.

By parts (i-ii) of Proposition 1, following a small reduction in firm  $i$ 's marginal cost, consumer surplus will always increase, although total profits will decrease if firm  $i$  is a high-cost firm. Total profits fall because the output of a high-cost firm increases at the expense of the outputs of more efficient firms. By part (iii) of Proposition 1, total welfare will decrease if firm  $i$ 's marginal cost is above the critical level  $c_i^{B**}$ . That is, if firm  $i$  is sufficiently inefficient the decrease in total profits will outweigh the increase in consumer surplus, resulting in a decrease in total welfare.<sup>6</sup>

Corollary 1 below shows that the effects of small cost reductions on performances also can be characterized using output shares. By (6), a firm's equilibrium output share is

$$t_i^B = \frac{q_i^B}{Q^B} = \frac{1}{n} + \frac{(1+\gamma)(2n+(n-1)\gamma)}{n(2n+(2n-1)\gamma)} \cdot \frac{(\bar{c}-c_i)}{(V-\bar{c})}. \quad (12)$$

Therefore, under Bertrand competition, firms' output shares have the reverse order of their marginal costs: the most efficient firm has the largest share, and the least efficient firm has the smallest share. It follows from (12) that at the Bertrand equilibrium all efficient firms have output shares greater than  $1/n$ , and all inefficient firms have output shares less than  $1/n$ .

**Corollary 1.** *Under Bertrand competition, a small reduction in firm  $i$ 's marginal cost*

(i) *reduces total profits  $\hat{U}$  its output share is below a critical level  $t_{\Pi}^B$ ;*

(ii) *reduces total welfare  $\hat{U}$  its output share is below a critical level  $t_w^B$ , where the*

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<sup>6</sup> The perverse welfare effect of cost reductions cannot arise in a homogenous Bertrand model, in which only the most efficient firm (firm 1) has a positive output in equilibrium. The equilibrium price is either firm 1's monopoly price (when it is below firm 2's marginal cost  $c_2$ ) or equal to  $c_2$ . In the former case, small reductions in  $c_j$  ( $j \neq 1$ ) have no effect, and a small reduction in  $c_1$  increases both consumer surplus and firm 1's profit (and therefore total welfare). In the latter case, a small reduction in  $c_1$  will not affect consumer surplus but will increase firm 1's profit (and therefore total welfare); small reductions in  $c_j$  ( $j \geq 3$ ) have no effect; and a small reduction in  $c_2$  will raise consumer surplus. Firm 1's profit will be lower but total welfare will be higher.

critical levels  $t_{\Pi}^B$  and  $t_W^B$  are given in (23) and (25) in the appendix.

Our results in Proposition 1 can be extended to other models of price competition in a straightforward manner. Here we briefly consider the logit model.<sup>7</sup> Firm  $i$ 's profit is  $\pi_i(p) = (p_i - c_i)s_i - K_i$ ,  $i = 1, \dots, n$ , where  $p_i$ ,  $c_i$  and  $K_i$  are  $i$ 's price, marginal cost, and sunk entry cost, respectively, and  $s_i$  is its market share or logit demand. As shown in Anderson and De Palma (2001), the Bertrand equilibrium is determined by  $(p_i - c_i) = \mu/(1 - s_i)$ , for all  $i$ , where  $\mu > 0$  is a demand parameter. Let  $p_j^*$  and  $s_j^*$  be  $j$ 's equilibrium price and market share, respectively, then the effect of a small reduction in  $c_i$  on welfare ( $W^* = \Pi^* + CS^*$ ) is

$$\frac{\partial W^*}{\partial c_i} = \frac{(1 - s_i^*)^3}{1 - s_i^* + (s_i^*)^2} + \frac{s_i^*(1 - s_i^*)^2}{(1 - \alpha)[1 - s_i^* + (s_i^*)^2]} \sum_{j=1}^n \frac{(s_j^*)^2}{1 - s_j^* + (s_j^*)^2} - 1, \text{ and}$$

$$\frac{\partial W^*}{\partial c_i} > 0 \Leftrightarrow \frac{(1 - s_i^*)^3}{1 - s_i^* + (s_i^*)^2} \left[ 1 - s_i^* + \frac{s_i^*}{(1 - \alpha)} \sum_{j=1}^n \frac{(s_j^*)^2}{1 - s_j^* + (s_j^*)^2} \right] > 1,$$

where  $\alpha = \sum_{j=1}^n [(s_j^*)^2 / (1 - s_j^* + (s_j^*)^2)]$  satisfies  $0 < \alpha < 1$ . The above inequality implicitly defines a critical value for  $i$ 's market share, below which its small cost reduction decreases welfare.

### 3. The Cournot Model

The inverse demand functions corresponding to the Bertrand-Shubik model (1) are:

$$p_i(q) = p_i(q_1, \dots, q_n) = V - \frac{1}{1 + \gamma} (q_i + \gamma \bar{q}) = V - q_i + \frac{\mathbf{g}}{1 + \mathbf{g}} (q_i - \bar{q}), \quad (13)$$

where  $\bar{q} = (\sum_{i=1}^n q_i) / n$  is the average quantity,  $V$  and  $\gamma$  are the same as in (1), and the quantity

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<sup>7</sup> Anderson et al. (1992) provided extensive coverage of both the logit model discussed here and the Hotelling model. Part 1 of Appendix B provides detailed derivations for the logit model. Note that the Bertrand-Shubik demand system (1) does not include the Hotelling model as a special case. However, one can easily show that similar results as in Proposition 1 hold for the Hotelling model. Part 2 of Appendix B derives such results.

space is  $\{q \in \mathbb{R}_+^n : V - q_i + \gamma(q_i - \bar{q}) / (1 + \gamma) > 0, i = 1, \dots, n\}$ . The first-order conditions for the firms' profit maximization can be arranged as

$$Cq = \begin{bmatrix} 2(1 + \frac{\gamma}{n}) & \frac{\gamma}{n} & \dots & \frac{\gamma}{n} \\ \frac{\gamma}{n} & 2(1 + \frac{\gamma}{n}) & \dots & \frac{\gamma}{n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\gamma}{n} & \frac{\gamma}{n} & \dots & 2(1 + \frac{\gamma}{n}) \end{bmatrix} \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix} = \begin{pmatrix} (1 + \gamma)(V - c_1) \\ (1 + \gamma)(V - c_2) \\ \vdots \\ (1 + \gamma)(V - c_n) \end{pmatrix}. \quad (14)$$

Solving (14) for  $q$  leads to the following Cournot equilibrium values:<sup>8</sup>

$$p_i^c = \frac{(n + \gamma)V}{2n + (n + 1)\gamma} + \frac{n\gamma(n + \gamma)\bar{c}}{(2n + \gamma)(2n + (n + 1)\gamma)} + \frac{nc_i}{2n + \gamma}; \quad (15)$$

$$q_i^c = \frac{n(1 + \gamma)}{n + \gamma}(p_i^c - c_i), \quad Q^c = \sum q_i^c = \frac{n^2(1 + \gamma)}{2n + (n + 1)\gamma}(V - \bar{c}); \text{ and} \quad (16)$$

$$\pi_i^c = \frac{n + \gamma}{n(1 + \gamma)}(q_i^c)^2. \quad (17)$$

Similar to the Bertrand model, we make the following assumption (A2), which guarantees a positive output for each firm at the above Cournot equilibrium:<sup>9</sup>

$$\text{A2: } \frac{(2n + g)V + g\sum_{j \neq n} c_j}{n(g + 2)} > c_n.$$

It is useful to note that A2 follows from A1, so positive outputs under Bertrand competition

<sup>8</sup> The inverse of matrix  $C$  can be obtained the same way as  $B^1$  (see footnote 3). Specifically,  $C^{-1} = (n/(2n + \gamma))\{I - [\gamma/(2n + (n + 1)\gamma)]H\}$ . Multiplying  $C^{-1}$  into the right hand side of (14) gives (15). Equations (16) and (17) follow directly by utilizing (15) and the inverse demand equations (13).

<sup>9</sup> Similar to A1 in the Bertrand model, A2 determines a feasible range  $(c_L^c, c_U^c)$  for each  $c_k$  in the Cournot model, where  $c_U^c = c_U^c(k)$  and  $c_L^c = c_L^c(k)$  are its upper and lower bounds as given below:

$$c_U^c = \frac{(2n + g)V + g\sum_{j \neq k} c_j}{n(g + 2)}, \text{ and } c_L^c = \begin{cases} \frac{[n(g + 2)c_n - ((2n + g)V + g\sum_{j \neq k, n} c_j)]^+}{g} & \text{if } k < n \\ \frac{[n(g + 2)c_{n-1} - ((2n + g)V + g\sum_{j=1}^{n-2} c_j)]^+}{g} & \text{if } k = n. \end{cases}$$

imply positive outputs under Cournot competition.

By (16),  $(p_i^c - c_i)/q_i^c = (n+\gamma)/(n(1+\gamma))$ , so all firms' markup/output ratios are the same. It is useful to note that the markup  $(p_i^c - c_i)$  is given by

$$p_i^c - c_i = \frac{(n+\gamma)(V - \bar{c})}{2n + (n+1)\gamma} + \frac{(n+\gamma)(\bar{c} - c_i)}{2n + \gamma},$$

so more efficient firms (i.e., with smaller  $c_i$ ) will have larger markups and outputs.

Differentiating (15)-(17) shows immediately that the effects of a small reduction in firm  $i$ 's marginal cost on prices, outputs and profits at the Cournot equilibrium are similar to those at the Bertrand equilibrium: (i) it decreases firm  $i$ 's price, and increases its output and profits; (ii) it decreases all other firms' prices, outputs and profits; and (iii) it increases the industry output.<sup>10</sup>

We next study the effect of small changes in costs on total profits as well as consumer surplus and total welfare in the Cournot model. Direct calculations show that, at the Cournot equilibrium, industry profits  $\Pi^c = \sum_{j=1}^n \pi_j^c$ , consumer surplus ( $CS^c$ ) and total welfare ( $W^c$ ), respectively, are given by

$$\Pi^c = \frac{n+\gamma}{n(1+\gamma)} \sum_{j=1}^n (q_j^c)^2, \quad (18)$$

$$CS^c = \frac{1}{2(1+\gamma)} \sum_{j=1}^n (q_j^c)^2 + \frac{\gamma}{2n(1+\gamma)} (Q^c)^2, \quad (19)$$

$$W^c = \Pi^c + CS^c = \frac{3n+2\gamma}{2n(1+\gamma)} \sum_{j=1}^n (q_j^c)^2 + \frac{\gamma}{2n(1+\gamma)} (Q^c)^2, \quad (20)$$

where  $q_j^c$  and  $Q^c$  are given by (16).

Proposition 2 characterizes the effects of small cost reductions under Cournot

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<sup>10</sup> See part 3 of Appendix B for closed form expressions of these effects.

competition.

**Proposition 2.** *Under Cournot competition, a small reduction in firm  $i$ 's marginal cost*

(i) *reduces total profits  $\hat{U}$  if  $c_i$  is above a critical level  $c_i^{C*}$ ;*

(ii) *always increases consumer surplus; and*

(iii) *reduces total welfare  $\hat{U}$  if  $c_i$  is above a critical level  $c_i^{C**}$ , where the critical levels  $c_i^{C*}$  and  $c_i^{C**}$  are given respectively by (28) and (30) in the appendix.*

It is straightforward to show that  $c_i^{C**} > c_i^{C*}$ . Hence, if a cost reduction increases total profits it also will increase total welfare. What is more interesting is the negative welfare effect: a firm's cost reduction decreases total welfare if its marginal cost is above the critical level  $c_i^{C**}$  in (30). The explanations for this counter-intuitive result are similar to those for the welfare effects under Bertrand competition discussed in Section 2.

The next corollary characterizes the welfare effects of a Cournot firm's small cost reductions using its output shares. From (16), firm  $i$ 's equilibrium output share is

$$t_i^C = \frac{q_i^C}{Q^C} = \frac{1}{n} + \frac{(2n + (n + 1)\gamma)}{n(2n + \gamma)} \cdot \frac{(\bar{c} - c_i)}{(V - \bar{c})}. \quad (21)$$

It follows immediately from (21) that output shares in the Cournot model are ranked similarly as in the Bertrand model: the most efficient firm has the largest share, and the least efficient firm has the smallest share. Moreover, all efficient firms have output shares greater than  $1/n$  and all inefficient firms have output shares less than  $1/n$ .

**Corollary 2.** *Under Cournot competition, a small reduction in firm  $i$ 's marginal cost*

(i) *reduces total profits  $\hat{U}$  if its output share is below a critical level  $t_{\Pi}^C$ ;*

(ii) *reduces total welfare  $\hat{U}$  if its output share is below a critical level  $t_w^C$ , where the critical levels  $t_{\Pi}^C$  and  $t_w^C$  are given in (27) and (29) in the appendix.*

#### 4. Bertrand vs. Cournot

By comparing our results in Sections 3 and 4, we obtain the following three differences between Bertrand and Cournot competition that are relevant to our comparative statics analysis: (I) total output under Cournot competition is lower than that under Bertrand competition; (II) a larger market share is required at Bertrand equilibrium than at Cournot equilibrium for a cost reduction to have a positive welfare effect; and (III) the distribution of output shares is wider under Bertrand competition than under Cournot competition.

By (6) and (16),  $Q^C - Q^B = -n(n-1)g^2(V - \bar{c})/\{(2n + (n-1)g)(2n + (n+1)g)\} < 0$ , so Claim I holds. By (25) and (29),

$$t_w^B - t_w^C = \frac{\gamma^2[3n^3 + (3n^2 - 1)n\gamma + (n^2 + 2n - 3)\gamma^2]}{(2n + (n-1)\gamma)(2n + (2n+1)\gamma)(3n + (3n-1)\gamma)(3n + 2\gamma)} > 0, \quad (22)$$

so Claim II holds. By (12) and (21), the difference between firm  $i$ 's output shares in Bertrand and Cournot equilibria is

$$t_i^B - t_i^C = \frac{(n-1)\gamma^3}{(2n + (2n-1)\gamma)(2n + \gamma)} \cdot \frac{(\bar{c} - c_i)}{(V - \bar{c})}.$$

This difference is positive (i.e.,  $t_i^B > t_i^C$ ) for all efficient firms and negative (i.e.,  $t_i^B < t_i^C$ ) for all inefficient firms. Hence, all efficient firms have larger output shares under Bertrand competition than under Cournot competition, and all inefficient firms have smaller output shares under Bertrand competition than under Cournot competition. Claim III also follows from an examination of the output shares within each model. By (12) and (21),

$$t_i^B - t_j^B = \frac{(1+\gamma)(2n + (n-1)\gamma)}{n(2n + (2n-1)\gamma)} \cdot \frac{(c_j - c_i)}{(V - \bar{c})}, \text{ and}$$

$$t_i^C - t_j^C = \frac{2n + (n+1)\gamma}{n(2n + \gamma)} \cdot \frac{(c_j - c_i)}{(V - \bar{c})}.$$

It follows from  $(1 + \gamma)(2n + (n - 1)\gamma)/(2n + (2n - 1)\gamma) > (2n + (n + 1)\gamma)/(2n + \gamma)$  that  $t_i^B - t_j^B > t_i^C - t_j^C$  for all  $i < j$  (recall the assumption that  $c_1 \leq c_2 \leq \dots \leq c_n$ ). Hence, the difference in output shares between any two firms is larger under Bertrand competition than under Cournot competition.

The above three differences lead to the next proposition.

**Proposition 3.** *For each inefficient firm  $i$  (i.e.,  $c_i > \bar{c}$ ), a small reduction in  $c_i$  will decrease welfare under Bertrand competition if it decreases welfare under Cournot competition.*

Example 1 below shows a case in which small cost reductions increase (decrease) welfare in Cournot (Bertrand) competition. Note that earlier works comparing Bertrand and Cournot equilibria showed that prices at Bertrand equilibria are intrinsically lower than at Cournot equilibria,<sup>11</sup> and that total welfare at Bertrand equilibria are higher than at Cournot equilibria in our model.<sup>12</sup>

**Example 1.** *Consider  $n = 3$  with  $V = 12$ ,  $(c_1, c_2, c_3) = (1, 4.5, 9.7)$  and  $\gamma = 0.8$ . The Bertrand and Cournot equilibria and their performances are:*

$$\begin{aligned} (p_1^B, p_2^B, p_3^B) &= (5.933, 7.543, 9.935), (p_1^C, p_2^C, p_3^C) = (6.136, 7.680, 9.975); \\ (\mathbf{p}_1^B, \mathbf{p}_2^B, \mathbf{p}_3^B) &= (37.311, 14.197, 0.085), (\mathbf{p}_1^C, \mathbf{p}_2^C, \mathbf{p}_3^C) = (37.490, 14.374, 0.107); \\ Q^B &= 12.590, Q^C = 12.209; CS^B = 33.715, CS^C = 31.556; W^B = 85.307, W^C = 83.527; \\ \mathbb{P}^B/c_3 &= 0.415 > 0, \mathbb{P}^C/c_3 = 0.375 > 0; \mathbb{W}^B/c_3 = 0.055 > 0, \mathbb{W}^C/c_3 = -0.015. \end{aligned}$$

*So a small cost reduction by firm 3 decreases total welfare under Bertrand competition but increases total welfare under Cournot competition. It reduces total profits in both cases.*

<sup>11</sup> See Singh and Vives (1984), Hackner (2000), and Amir and Jin (2001).

<sup>12</sup> Hsu and Wang (2005) compared welfare at Bertrand and Cournot equilibria in a differentiated goods oligopoly. Although their focus is on the Hackner (2000) model with quality differentiation and identical costs, their results can be generalized in a straightforward manner to show that in the present model with unequal costs both consumer surplus and total welfare are higher under Bertrand competition than under Cournot competition.

## 5. Conclusion and Extensions

This paper has provided the closed-form expressions for the critical levels of output shares below which a small cost reduction by a Bertrand or Cournot firm reduces welfare in linear oligopolies with differentiated goods. It has shown that the critical value is larger under Bertrand competition than under Cournot competition.

In addition to extending our analysis to Bertrand oligopolies with non-linear demand or non-linear costs or both, we suggest three other closely related future studies. First, it would be useful to evaluate the welfare effects of shocks under Bertrand competition, which are extensions of recent works under Cournot competition, such as Salant and Shaffer (1999), Long and Soubeyran (2001), Linnemer (2003), and Février and Linnemer (2004), that studied such effects under Cournot competition. Second, it would be useful to extend the profit effects of taxation policies in a symmetric Bertrand model seen in Anderson, de Palma, and Kreider (2001) to the welfare effects of taxation policies in symmetric or asymmetric Bertrand models. Third, it would be useful to extend the welfare effects of the trade policies seen in Lahiri and Ono (1997), Leahy and Montagna (2001), and Neary (2002) to our differentiated Bertrand or Cournot model.

## Appendix

**Proof of Proposition 1 and Corollary 1:** (i) By (5) and (6), differentiating (9) gives

$$\begin{aligned} \frac{\partial \Pi^B}{\partial c_i} &= \frac{2n}{n + (n-1)\mathbf{g}} \sum_{j=1}^n q_j^B \frac{\partial q_j^B}{\partial c_i} = 2 \left\{ \sum_{j=1}^n q_j^B \frac{\partial p_j^B}{\partial c_i} - q_i^B \right\} \\ &= \frac{2n(1+\mathbf{g})}{2n + (2n-1)\mathbf{g}} \left[ \frac{\mathbf{g}(n + (n-1)\mathbf{g})}{n(1+\mathbf{g})(2n + (n-1)\mathbf{g})} Q^B - q_i^B \right] \end{aligned}$$

$$= \frac{2n(1+\gamma)Q^B}{2n+(2n-1)\gamma} \left[ \frac{\gamma(n+(n-1)\gamma)}{n(1+\gamma)(2n+(n-1)\gamma)} - t_i^B \right],$$

which leads to

$$\begin{aligned} \frac{\partial \Pi^B}{\partial c_i} < 0 &\Leftrightarrow t_i^B > t_{\Pi}^B = \frac{\mathbf{g}(n+(n-1)\mathbf{g})}{n(1+\mathbf{g})(2n+(n-1)\mathbf{g})} \\ &\Leftrightarrow c_i - \bar{c} > \frac{(2n+(2n-1)\gamma)^2(V-\bar{c})}{(1+\gamma)^2(2n+(n-1)\gamma)^2}, \end{aligned} \quad (23)$$

which is equivalent to  $c_i > c_i^{B*}$ , where

$$c_i^{B*} = \frac{n(2n+(2n-1)\gamma)^2 V + \gamma(n+(n-1)\gamma)((n-1)\gamma^2 + (5n-2)\gamma + 4n) \sum_{j \neq i} c_j}{(n-1)(1+\gamma)^2(2n+(n-1)\gamma)^2 + (2n+(2n-1)\gamma)^2}. \quad (24)$$

(ii) Differentiating (10) with respect to  $c_i$  gives

$$\begin{aligned} \frac{\partial CS^B}{\partial c_i} &= \frac{1}{1+\gamma} \sum_{j=1}^n q_j^B \frac{\partial q_j^B}{\partial c_i} + \frac{\gamma}{n(1+\gamma)} Q^B \frac{\partial Q^B}{\partial c_i} \\ &= - \frac{\gamma(n+(n-1)\gamma)}{(2n+(n-1)\gamma)(2n+(2n-1)\gamma)} Q^B - \frac{(n+(n-1)\gamma)}{2n+(2n-1)\gamma} q_i^B < 0. \end{aligned}$$

(iii) Finally, utilizing the above results leads to

$$\begin{aligned} \frac{\partial W^B}{\partial c_i} &= \frac{\partial \Pi^B}{\partial c_i} + \frac{\partial CS^B}{\partial c_i} = \frac{\mathbf{g}(n+(n-1)\mathbf{g})}{(2n+(n-1)\mathbf{g})(2n+(2n-1)\mathbf{g})} Q^B - \frac{(3n+(3n-1)\mathbf{g})}{2n+(2n-1)\mathbf{g}} q_i^B \\ &= \frac{(3n+(3n-1)\gamma)Q^B}{2n+(2n-1)\gamma} \left[ \frac{\gamma(n+(n-1)\gamma)}{(2n+(n-1)\gamma)(3n+(3n-1)\gamma)} - t_i^B \right], \end{aligned}$$

which leads to

$$\begin{aligned} \frac{\partial W^B}{\partial c_i} < 0 &\Leftrightarrow t_i^B > t_W^B = \frac{\gamma(n+(n-1)\gamma)}{(2n+(n-1)\gamma)(3n+(3n-1)\gamma)} \\ &\Leftrightarrow c_i - \bar{c} > \frac{(2n+(2n-1)\gamma)^2(3n+(n-1)\gamma)(V-\bar{c})}{(1+\gamma)(2n+(n-1)\gamma)^2(3n+(3n-1)\gamma)}, \end{aligned} \quad (25)$$

which is equivalent to  $c_i > c_i^{B**}$ , where  $c_i^{B**}$  is equal to

$$\frac{n(2n+(2n-1)\gamma)(3n+(n-1)\gamma)V + \gamma(n+(n-1)\gamma)((3n^2-4n+1)\gamma^2 + 11n^2\gamma - 6n\gamma + 8n^2) \sum_{j \neq i} c_j}{(n-1)(1+\gamma)(2n+(n-1)\gamma)^2(3n+(3n-1)\gamma) + (2n+(2n-1)\gamma)^2(3n+(n-1)\gamma)}. \quad (26)$$

**Q.E.D.**

**Proof of Proposition 2 and Corollary 2:** (i) By (15) and (16), differentiating (18) gives

$$\begin{aligned} \frac{\partial \Pi^C}{\partial c_i} &= \frac{2(n+g)}{n(1+g)} \sum_{j=1}^n q_j^C \frac{\partial q_j^C}{\partial c_i} = \frac{2(n+g)}{2n+g} \left[ \frac{g}{2n+(n+1)g} Q^C - q_i^C \right] \\ &= \frac{2(n+g)Q^C}{2n+g} \left[ \frac{g}{2n+(n+1)g} - t_i^C \right], \end{aligned}$$

which leads to

$$\begin{aligned} \frac{\partial \Pi^C}{\partial c_i} > 0 &\Leftrightarrow t_i^C < t_{\Pi}^C = \frac{\gamma}{2n+(n+1)\gamma} \\ &\Leftrightarrow c_i - \bar{c} > \frac{(2n+\gamma)^2(V-\bar{c})}{(2n+(n-1)\gamma)^2}, \end{aligned} \quad (27)$$

which is equivalent to  $c_i > c_i^{C*}$ , where

$$c_i^{C*} = \frac{n(2n+\gamma)^2V + (n(2n+(n+1)\gamma)^2 - (2n+\gamma)^2) \sum_{j \neq i} c_j}{(n-1)(2n+(n+1)\gamma)^2 + (2n+\gamma)^2}. \quad (28)$$

(ii) Differentiating (19) with respect to  $c_i$  gives

$$\begin{aligned} \frac{\partial CS^C}{\partial c_i} &= \frac{1}{1+\gamma} \sum_{j=1}^n q_j^C \frac{\partial q_j^C}{\partial c_i} + \frac{\gamma}{n(1+\gamma)} Q^C \frac{\partial Q^C}{\partial c_i} \\ &= - \frac{\gamma(n+\gamma)}{(2n+\gamma)(2n+(n+1)\gamma)} Q^C - \frac{n}{2n+\gamma} q_i^C < 0. \end{aligned}$$

(iii) Finally, utilizing the above results

$$\begin{aligned} \frac{\partial W^C}{\partial c_i} &= \frac{\partial \Pi^C}{\partial c_i} + \frac{\partial CS^C}{\partial c_i} = \frac{\gamma(n+\gamma)}{(2n+\gamma)(2n+(n+1)\gamma)} Q^C - \frac{3n+2\gamma}{2n+\gamma} q_i^C \\ &= \frac{(3n+2\gamma)Q^C}{2n+\gamma} \left[ \frac{\gamma(n+\gamma)}{(2n+(n+1)\gamma)(3n+2\gamma)} - t_i^C \right], \end{aligned}$$

which leads to

$$\begin{aligned} \frac{\partial W^C}{\partial c_i} > 0 &\Leftrightarrow t_i^C < t_W^C = \frac{\gamma(n+\gamma)}{(2n+(n+1)\gamma)(3n+2\gamma)} \\ &\Leftrightarrow c_i - \bar{c} > \frac{(2n+\gamma)^2(3n+(n+2)\gamma)}{(2n+(n+1)\gamma)^2(3n+\gamma)}(V-\bar{c}), \end{aligned} \quad (29)$$

which is equivalent to  $c_i > c_i^{C^{**}}$ , where  $c_i^{C^{**}}$  is equal to

$$\frac{n(2n+\mathbf{g})^2(3n+(n+2)\mathbf{g})V + \mathbf{g}(8n^3 - 4n^2 + 3n^3\mathbf{g} + 6n^2\mathbf{g} - 4n\mathbf{g} + n^2\mathbf{g}^2 + n\mathbf{g}^2 - \mathbf{g}^2) \sum_{j \neq i} c_j}{(n-1)(2n+(n+1)\mathbf{g})^2(3n+\mathbf{g}) + (2n+\mathbf{g})^2(3n+(n+2)\mathbf{g})}. \quad (30)$$

**Q.E.D.**

**Proof of Proposition 3:** Using the fact that  $t_i^B < t_i^C$  for inefficient firms, by (22), Corollary 1

and Corollary 2,  $\partial W^C/\partial c_i > 0 \Rightarrow t_i^C < t_W^C \Rightarrow t_i^B < t_i^C < t_W^C < t_W^B \Rightarrow \partial W^B/\partial c_i > 0$ . **Q.E.D.**

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