

The Core and Coalition Formation in an N -person Network Game*

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Summary: Network games are truncated coalitional games whose values are only defined for a subset of coalitions. This paper first provides three equivalent conditions for a non-empty old TU core: sub-balancedness, minimum no-blocking payoff ($mnbp \leq v(N)$), and maximal payoff ($mp \leq v(N)$). It then shows that players will achieve the mp by forming optimal and minimal sub-balanced collections according to the unique balancing vector, and they will split mp within the old core if old core is non-empty and within the optimal set of $mnbp$ if old core is empty. Finally, it obtains analogous results for NTU network games.

Keywords: Coalition formation, core, maximal payoff, minimum no-blocking payoff, network game.

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1. Introduction

This paper studies n -person network games, which are truncated coalitional games in which the values or payoffs are defined only for a given set of coalitions. In other words, an n -person network game is a set function (or a set correspondence) whose domain is a given collection of coalitions among the n players and whose range is the non-negative half space (or spaces). It is useful to note that in such network games the grand coalition's payoff is not necessarily defined, so network games are more general than coalitional games, which are special network games in which the values are defined for all coalitions.

The paper first studies the old core of a transferable utilities (TU) network game in which the grand coalition's payoff $v(N)$ is defined. The old TU core is defined as the set of splits of $v(N)$ that are unblocked by each feasible coalition in the game. By investigating three different linear programming problems arising from the game, the paper establishes three equivalent necessary and sufficient conditions for a non-empty old TU core: *i*) the n -person network game is sub-balanced; *ii*) the game's minimum no-blocking payoff (*mnbp*) is no greater than $v(N)$; and *iii*) the game's maximal payoff (*mp*) is no greater than $v(N)$. These core results are non-trivial extensions of the known results in coalitional TU games to TU network games: condition *i*) above is an extension of the famous Bondareva-Shapley theorem (Bondareva [1962], Shapley [1967]), and conditions *ii*) and *iii*) are extensions of the author's earlier results (Zhao [2001] and [2006]).

The above condition *iii*) reveals that when the old core is empty, players will be able to generate the *mp* that is greater than the grand coalition's payoff $v(N)$. Our exploration of such generated and better payoffs leads to four more important questions: What payoffs will be split? How will the payoff be split? What coalitions will be formed? and How long will

each of the coalitions be formed in coalitional TU network games? Briefly answering these questions (in order), players will split the game's mp ; they will always split mp within the new core, which coincides with the old core when the old core is non-empty and the optimal set of mbp when the old core is empty; they will form optimal and minimal sub-balanced collections that generate the game's mp ; and they form each coalition in the formed collection for a length or percentage of time determined by the collection's unique balancing vector.

Finally, the paper obtains analogous results in non-transferable utilities (NTU) network games. Due to the generality of non-transferable utilities, some of the NTU results are weaker than the corresponding TU results. What is striking is that new coalitions, though not originally defined in the network game, could be formed as the union of players belonging to minimal sub-balanced collections that generate the game's maximal payoff.

The rest of the paper is organized as follows. Section 2 defines the old TU core and the minimum no-blocking payoff, section 3 establishes three existence theorems on non-empty old core, and section 4 shows that the new core is always non-empty and completes the coalition formation theory in TU network games. Section 5 obtains analogous results in NTU network games, section 6 concludes, and the appendix provides the proofs.

2. Description of the Problem

Let $N = \{1, 2, \dots, n\}$ be the set of players, $\mathcal{N} = 2^N$ be the set of all subsets (also called coalitions) of N , and ω be a proper subset of \mathcal{N} (i.e., $\omega \subset \mathcal{N}$). Then, each set function $v: \omega \rightarrow \mathbf{R}_+$ with $v(\emptyset) = 0$, is called an n -person network game, denoted as

$$(1) \quad \Gamma_\omega = \{N, \omega, v(\cdot)\},$$

which specifies a joint payoff $v(S)$ for each coalition $S \in \omega$. Note that the above game does

not require $N \in \omega$, so grand coalition's payoffs is not necessarily defined in a network game. It is straightforward to see that (1) becomes a coalitional TU game (also called TU game in characteristic form) when $\omega = \mathcal{N}$, so network games are truncated coalitional TU games, which are obtained by restricting a coalitional game to each proper subset of \mathcal{N} . Alternatively, a network game can be understood as an *incomplete graph* with nonlinear payoffs, in the sense that a coalitional game is equivalent to a *complete graph* with nonlinear payoffs in which players are the nodes of the graph and $v(S)$ are the payoffs for each $S \in \mathcal{N}$.

We first define the old core, which assumes $N \in \omega$ and studies the question of how to split the grand coalition's payoff $v(N)$. Let the set of all splits of $v(N)$ be denoted as $X(v(N)) = \{x \in \mathbf{R}_+^n \mid \sum_{i \in N} x_i = v(N)\}$, where for each $x = (x_1, \dots, x_n) \in X(v(N))$ and $i \in N$, x_i is player i 's payoff or share of $v(N)$. Given $S \in \omega$, a split $x \in X(v(N))$ is blocked by S if it gives S less than $v(S)$ (i.e., $\sum_{i \in S} x_i < v(S)$), and it is in the old core of the game (1) if it is unblocked by all $S \in \omega \setminus N$. Let such old core be denoted as

$$(2) \quad c_0 = c_0(\Gamma_\omega) = \{x \in X(v(N)) \mid \sum_{i \in S} x_i \geq v(S) \text{ for all } S \in \omega \setminus N\}.$$

The next section establishes three necessary and sufficient arguments for a non-empty old core. The first argument is that the above core is nonempty if and only if the network game is sub-balanced (see next section for formal definition). This result is obtained by applying the duality theorem of the following linear programming problem:

$$(3) \quad \text{Min } \{\sum_{i \in N} x_i \mid x \in \mathbf{R}_+^n; \sum_{i \in S} x_i \geq v(S) \text{ for all } S \in \omega \setminus N, \text{ and } \sum_{i \in N} x_i = v(N)\},$$

which is analogous to the proof of Bondareva-Shapley theorem for coalitional TU games (Bondareva [1962], Shapley [1969]; see also Kannai [1992] and Myerson [1991] for simpler proofs).

Two other necessary and sufficient core arguments follow from a different linear

programming problem given below:

$$(4) \quad mnbp = mnbp(\Gamma_\omega) = \text{Min} \{ \sum_{i \in N} x_i \mid x \in \mathbf{R}_+^n; \sum_{i \in S} x_i \geq v(S) \text{ for all } S \in \omega \setminus N \},$$

whose optimal value is called the minimum no-blocking payoff (*mnbp*), which leads to our second core argument: the old core (the interior of old core) is nonempty if and only if $v(N)$ is no less than (greater than) *mnbp*, which is analogous the author's result for coalitional TU games (Zhao [2001]).

Similar to the concept of maximal payoff and coalition formation theory in coalitional TU games (Zhao [2006]), the duality theorem for (4) provides another argument for a non-empty old core, and it also answers four perhaps more important questions: What payoffs will be split? How will the payoff be split? What coalitions or sub-networks will form? and How long will each of these coalitions be formed by rational players in the network game (1)? These questions are answered in Section 4.

3. Three Necessary and Sufficient Conditions for a Non-empty Old Core

I first define the concept of sub-balancedness. Given a collection of coalitions $\mathcal{B} = \{T_1, \dots, T_k\} \subset \omega$, let $T = \cup_{i=1}^k T_i$. For each $i \in T$, let $\mathcal{B}(i) = \{S \in \mathcal{B} \mid i \in S\}$ be the set of coalitions containing i . Then, \mathcal{B} is a sub-balanced collection (or sub-balanced) if it has a balancing vector w (i.e., $w \in \mathbf{R}_{++}^k$ or $w_S > 0, S \in \mathcal{B}$) such that $\sum_{S \in \mathcal{B}(i)} w_S = 1$ for each $i \in T = \cup_{i=1}^k T_i$, and our network game (1) is sub-balanced if for each sub-balanced collection \mathcal{B} with its balancing vector w , $\sum_{S \in \mathcal{B}} w_S v(S) \leq v(N)$ holds. If $T = N$, sub-balancedness is the same as balancedness defined in Bondareva (1962), Scarf (1967) and Shapley (1969).

Similar to the author's interpretation of balanced collections (Zhao [2006]), a sub-balanced \mathcal{B} with a balancing vector w can be treated as a sub-balanced internet assignment

problem (\mathcal{B}, w) , in which students in $T = \cup_{i=1}^k T_i$ are assigned into k chat rooms and each chat room $S \in \mathcal{B}$ opens $100 \times w_S$ minutes, such that the participation time for each student is precisely 100 minutes.

The dual problem of the minimization problem (4) for *mbp* leads to the concept of generated payoffs as defined in Definition 1 below. A sub-balanced collection is minimal if it has no sub-balanced proper subset with the same union. Precisely, \mathcal{B} is a minimal sub-balanced collection if it is sub-balanced and there is no sub-balanced $\mathcal{B}' \subsetneq \mathcal{B}$ such that $\mathcal{B}' \neq \mathcal{B}$ and $\cup_{S \in \mathcal{B}'} S = \cup_{S \in \mathcal{B}} S = T$. Similar to the property of a minimal balanced collection, one can show that a sub-balanced collection is minimal if and only if its balancing vector is unique.

Definition 1: *Given the network game (1), let*

$$(5) \quad B = \{\mathcal{B} = \{T_1, \dots, T_k\} \subset \omega \mid N \notin \mathcal{B}, \mathcal{B} \text{ is a minimal sub-balanced collection}\}$$

denote the set of all minimal sub-balanced collections that are not $\{N\}$. For each $\mathcal{B} \in B$ with its unique balancing vector w , the payoff generated by \mathcal{B} is given by

$$(6) \quad gp(\mathcal{B}) = \sum_{S \in \mathcal{B}} w_S v(S),$$

and the maximum of generated payoffs (m gp) is given by

$$(7) \quad mgp = mgp(\Gamma_\omega) = \text{Max} \{gp(\mathcal{B}) \mid \mathcal{B} \in B\}.$$

Note that the set B in (5) includes each single coalition collection $\{S\} \neq \{N\}$ and each partition of all $S \in \omega \setminus N$. The definition considers only minimal sub-balanced collections, because adding non-minimal ones to the feasible set B will not affect the optimal value of (7), which is equivalent to the dual problem of the min problem (4). Let the maximal set for the above max problem (7) be denoted as

$$(8) \quad B_0 = B_0(\Gamma_\omega) = \{\mathcal{B} \in B \mid gp(\mathcal{B}) = mgp\} = \text{Arg-Max} \{gp(\mathcal{B}) \mid \mathcal{B} \in B\}.$$

For each maximal collection $\mathcal{B} \in B_0$ with its unique balancing vector w , players in $T = \cup_{S \in \mathcal{B}} S$

can generate the game's mgp by forming each $S \in \mathcal{B}$ for w_S units (or percentage) of the time.

An inessential game is one in which $v(N) < \sum_{i \in N} v(i)$ ¹. It is clear that rational players will not split $v(N)$ in such games because the payoff generated by the finest partition $\Delta_0 = \{\{1\}, \dots, \{n\}\}$ is greater than $v(N)$.

Similarly, rational players will not split $v(N)$ in games in which there is a partition Δ whose generated payoff is greater than $v(N)$ (i.e., $gp(\Delta) = \sum_{S \in \Delta} v(S) > v(N)$). Moving further along this line of argument, one concludes that it is irrational to split $v(N)$ if and only if the maximal generated payoff given in (7) is greater than $v(N)$, or if and only if the old core is empty, as implied in part iii) of Theorem 1 given below:

Theorem 1: *Given the network game (1), let its old core, $mnbp$ and mgp be given in (2), (4) and (7) respectively. Then, the following four claims are equivalent to each other:*

- (i) *the old core is non-empty;*
- (ii) *the network game is sub-balanced;*
- (iii) *the minimum no-blocking payoff is below $v(N)$; and*
- (iv) *the maximal generated payoff is below $v(N)$.*

Theorem 1 is proved in the appendix. To put it differently, Theorem 1 provides three necessary and sufficient arguments for an empty old core in TU network games: the game is not sub-balanced, $v(N)$ is below $mnbp$, and it is irrational to split $v(N)$.

The next section proposes a solution for network games whose core is empty.

4. The New Core and Coalition Formation in TU Network Games

The previous section shows that rational players will not split $v(N)$ in network games with $N \in \omega$ and with an empty old core. Then, what payoffs will rational players split in such

¹ We simplify $v(\{i\})$ as $v(i)$, $v(\{1,2\})$ as $v(12)$. Similar simplifications apply to other coalitions.

games with an empty old core? More generally, what payoffs will rational players split in coalitional TU network games in which $v(N)$ is not defined? We propose that they will split the maximal payoff defined below:

Definition 2: The maximal payoff (mp) for game (1) is given by

$$(9) \quad mp = mp(\Gamma_\omega) = \begin{cases} \text{Max} \{mgp; v(N)\} & \text{if } N \in \omega; \\ mgp & \text{if } N \notin \omega; \end{cases}$$

where $mgp = mgp(\Gamma_\omega)$ is the maximum of generated payoffs given in (7).

It is clear that rational players will split $mp = mgp$ in network games with $N \notin \omega$. In games with $N \in \omega$, it is rational to split $v(N) = mp$ when $c_0(\Gamma_\omega) \neq \emptyset$, and $mgp = mp > v(N)$ when $c_0(\Gamma_\omega) = \emptyset$. Hence, rational players will always split the game's maximal payoff given in (9), and this answers the question of what payoffs will be split.

Next, consider the question of how to split the game's mp . A split of mp is stable if no minimal sub-balanced collection \mathcal{B} can block it, because each split of mp is unblocked by N and blocking by each $S \in \omega \setminus N$ is included in blockings by B . This leads to the following new core. Let B and $gp(\mathcal{B})$ be given in (5) and (6) respectively. For each $\mathcal{B} \in \mathcal{B}$, let $T(\mathcal{B}) = \cup_{S \in \mathcal{B}} S$ denote the coalition of its players. Then the core or the new core of (1) can be defined as

$$(10) \quad c = c(\Gamma_\omega) = \{x \in X(mp) \mid \sum_{i \in T(\mathcal{B})} x_i \geq gp(\mathcal{B}) \text{ for all } \mathcal{B} \in \mathcal{B}\}.$$

In words, the new core is the subset of splits of the maximal payoff that is unblocked by all minimal sub-balanced collections.

The above new core is closely related to the optimal set for $mnbp$ in (4) given below:

$$(11) \quad Y = Y(\Gamma_\omega) = \text{Arg-Min} \{ \sum_{i \in N} x_i \mid x \in \mathbf{R}_+^n, \sum_{i \in S} x_i \geq v(S) \text{ for all } S \in \omega \setminus N \},$$

which is the set of splits of mgp (i.e., $\sum x_i = mnbp = mgp$).

Because each $x \in Y(\Gamma_\omega)$ satisfies the rationality for all $S \in \omega \setminus N$ and all $\mathcal{B} \in \mathcal{B}$, it possibly

can be blocked only by the grand coalition N . With $N \notin \omega$, each $x \in Y(\Gamma_\omega)$ is clearly stable against all deviations. With $N \in \omega$, the stability of each $x \in Y(\Gamma_\omega)$ falls into three cases:

Case 1. $mgp > v(N)$, or $c_0(\Gamma_\omega) = \emptyset$. In this case, x is stable against all deviations, because no $S \in \omega \cup N$ or $B \in B$ can block it.

Case 2. $mgp = v(N)$, or $c_0(\Gamma_\omega) \neq \emptyset$ and $Int c_0(\Gamma_\omega) = \emptyset$, where $Int c_0(\Gamma_\omega)$ is the (relative) interior of the old core. In this case, x is stable against all deviations, because $Y(\Gamma_\omega) = c_0(\Gamma_\omega)$.

Case 3. $mgp < v(N)$, or $Int c_0(\Gamma_\omega) \neq \emptyset$. In this case, x is clearly unstable because it violates the grand coalition's rationality (i.e., $\sum x_i = mgp < v(N)$).

The above discussions indicate that the new core is equal to the optimal set $Y(\Gamma_\omega)$ if $N \notin \omega$ or if $N \in \omega$ and $mgp > v(N)$, and the old core if $N \in \omega$ and $mgp \leq v(N)$.

Finally, consider the question of what coalitions will be formed. Because rational players will split the game's mp , coalitions formed by rational players should support the mp . By above discussions, rational players will form the grand coalition if $N \in \omega$ and $v(N) > mgp$, the minimal sub-balanced collections in B_0 in (8) if $N \notin \omega$ or if $N \in \omega$ and $v(N) < mgp$, and the union of N and B_0 if $N \in \omega$ and $v(N) = mgp$. The unique balancing vector for the formed minimal collection answers the question of how long will each of these coalitions be formed.

The next theorem summarizes the above answers.

Theorem 2: Given network game (1), let $mgp = mgp(\Gamma_\omega)$, $mp = mp(\Gamma_\omega)$ and $c = c(\Gamma_\omega)$ be given in (7), (9) and (10), respectively. Let $B^*(\Gamma_\omega) \neq \emptyset$ denote the set of optimal and stable coalitions that will be formed. Then, the following three claims hold:

- (i) rational players will split the maximal payoff mp ;
- (ii) the new core is always non-empty and is equal to

$$(12) \quad c(\Gamma_\omega) = \begin{cases} c_0 & \text{if } N \in \omega \text{ and } v(N) = mp; \\ Y & \text{if } N \notin \omega \text{ or if } N \in \omega \text{ and } v(N) < mp; \end{cases}$$

where $c_0 = c_0(\Gamma_\omega)$ and $Y = Y(\Gamma_\omega)$ are given respectively in (2) and (11); and

(iii) the set of optimal and stable coalitions that will be formed is given by

$$(13) \quad B^*(\Gamma_\omega) = \begin{cases} \{N\} & \text{if } N \in \omega \text{ and } v(N) > mgp; \\ \{N\} \cup B_0 & \text{if } N \in \omega \text{ and } v(N) = mgp; \\ B_0 & \text{if } N \notin \omega \text{ or if } N \in \omega \text{ and } v(N) < mgp; \end{cases}$$

where $B_0 = B_0(\Gamma_\omega)$ is given in (8); and for each $B \in B^*(\Gamma_\omega)$ with its unique balancing vector w , each coalition $T \in B$ will be formed for w_T unit (or percentage) of the time.

Observe that $c(\Gamma_\omega) \neq \emptyset$ always holds, so there always exists a split of the maximal payoff that is unblocked by any coalition or any sub-balanced collection. It is useful to compare the new core $c(I)$ in (10) with the old core $c_0(I)$ in (2). In the old core, players split $v(N)$ and only rule out deviations by each feasible coalition, whereas in the new core, players split the maximal payoff and rule out not only deviations by each feasible coalition, but also simultaneous deviations by each minimal sub-balanced collection of coalitions.

5. Extension to NTU Network Games

This section answers the questions of what subset of payoffs from which players will choose, how players choose a payoff vector, what coalitions will form, and how long each of these coalitions will be formed in NTU network games. Due to non-transferable utilities, some of these NTU results are weaker than the corresponding TU results. In particular, conditions for a non-empty old NTU core are only sufficient but not necessary.

An n-person NTU network game, or an NTU network game in characteristic form, is defined as

$$(14) \quad \Gamma_\omega = \{N, \omega, V(\cdot)\},$$

which specifies a non-empty set of payoffs, $V(S) \subset \mathbf{R}^S$, for each $S \in \omega$, where \mathbf{R}^S is the Euclidean space whose dimension is the number of players in S and whose coordinates are

the players in S . We use an uppercase V in $V(S)$ to denote NTU payoff (and an uppercase C in $C_0(\Gamma_\omega)$ to denote the old NTU core of (14) as given in (15)). For each $S \in \omega$, let the (weakly) efficient set of $V(S)$ be given as

$$\partial V(S) = \{ y \in V(S) \mid \text{there is no } x \in V(S) \text{ such that } x \gg y \},$$

where vector inequalities are defined as below: $x \geq y \Leftrightarrow x_i \geq y_i$, all i ; $x > y \Leftrightarrow x \geq y$ and $x \neq y$; and $x \gg y \Leftrightarrow x_i > y_i$, all i .

Similar to Scarf (1967b), the following two assumptions for (14) are assumed: (i) each $V(S)$ is closed and comprehensive (i.e., $y \in V(S)$, $u \in \mathbf{R}^S$ and $u \leq y$ imply $u \in V(S)$); (ii) for each S , $\{y \in V(S) \mid y_i \geq \partial V(i) > 0, \text{ all } i \in S\}$ is non-empty and bounded. It is useful to note that $\partial V(i) = \text{Max} \{x_i \mid x_i \in V(i)\}$. Under these assumptions, each $\partial V(S)$ is closed, non-empty, and bounded from above.

We first study the old core of (14) which is an extension of Scarf's core for coalitional NTU games (1967b). Given $S \in \omega$, a payoff vector $u \in \mathbf{R}_+^n$ is blocked by S if there is $y \in V(S)$ such that $y \gg u_S$ (i.e., $u_S \in V(S) \setminus \partial V(S)$), or in words, if S can obtain a higher payoff for each of its members than that given by u . The old core of (14) assumes $N \in \omega$ and is defined as

$$(15) \quad C_0 = C_0(\Gamma_\omega) = \{ u \in \partial V(N) \mid u_S \notin V(S) \setminus \partial V(S), \text{ all } S \in \omega \setminus N \}.$$

In words, a payoff vector $u \in \partial V(N)$ is in the old core if it is unblocked by all $S \in \omega \setminus N$. We now define the concept of a sub-balanced NTU network game geometrically. For each $S \in \omega \setminus N$ and $S \subset T$, let the number of players in S and T be $|S| = k < |T| = t$. Let $\tilde{v}_T(S) = V(S) \times \mathbf{R}^{T \setminus S} \subset \mathbf{R}^T$ denote the t -dimensional cylinder with $V(S)$, where $\mathbf{R}^{T \setminus S} = \prod_{i \in T \setminus S} \mathbf{R}^i$. Then, the sets of t -dimensional and n -dimensional payoffs generated by each minimal sub-balanced

\mathcal{B} with $T = \cup_{S \in \mathcal{B}} S$, and the set of generated payoffs can be defined as below:

Definition 3: Given a minimal sub-balanced $\mathcal{B} \in \mathcal{B}$ with $T = \cup_{S \in \mathcal{B}} S$, the t - and n -dimensional payoffs generated by \mathcal{B} and the set of generated payoffs of (14) are given, respectively, as

$$(16) \quad GP_T(\mathcal{B}) = \bigcap_{S \in \mathcal{B}} \tilde{v}(S) \subset \mathbf{R}^T,$$

$$(17) \quad GP(\mathcal{B}) = \bigcap_{S \in \mathcal{B}} \tilde{v}(S) \times \emptyset^{N \setminus T} \subset \mathbf{R}^n, \text{ and}$$

$$(18) \quad GP = GP(\Gamma_\omega) = \bigcup_{\mathcal{B} \in \mathcal{B}} GP(\mathcal{B}),$$

where $\emptyset^{N \setminus T} = \prod_{i \in N \setminus T} \emptyset^i$ is the empty complement space of \mathbf{R}^T , and \mathcal{B} is the set of minimal sub-balanced collections (excluding N) given in (5).

Note that (16) becomes $GP_T(\mathcal{B}) = \prod_{S \in \mathcal{B}} V(S)$ when \mathcal{B} is a partition. Similar to the TU case, (18) covers only minimal sub-balanced collections because adding non-minimal ones will not affect the efficient frontier of generated payoffs. It is important to keep in mind that the set of generated payoffs in (18) contain the projections of n -dimensional vectors on various subspaces, or those generated vector with empty components. It is possible, depending on the feasible set of coalitions ω , that all the generated payoff vectors in $GP(\Gamma_\omega)$ are projections on subspaces or that all the vectors in $GP(\Gamma_\omega)$ have some empty components.

Now, the NTU network game (14) is sub-balanced if

$$(19) \quad GP(\Gamma_\omega) \subset V(N)$$

holds, where $GP(\Gamma_\omega)$ is the generated payoffs in (18), or in words, (14) is a sub-balanced network game if for each sub-balanced \mathcal{B} , $u \in V(N)$ must hold if $u_S \in V(S)$ for all $S \in \mathcal{B}$. To understand a sub-balanced game geometrically, visualize that one is flying above the Rocky Mountains, and treat the generated payoffs as peaks of the mountains and $V(N)$ as clouds. Then, a game is sub-balanced if one sees only clouds (i.e., $GP(\Gamma_\omega) \subset V(N)$) and not sub-

balanced if one sees at least one peak above the clouds (i.e., $GP(\Gamma_\omega) \not\subset V(N)$).

Definition 4 below extends the concept of *mbp* in (4) to minimum no-blocking frontier (*MNBF*), and *mgp* in (7) to (weakly) efficient generated-payoffs (*EGP*). Recall that a payoff vector u is unblocked by S if $u_S \notin V(S) \setminus \partial V(S)$ or if $u \in [V(S) \setminus \partial V(S)]^C \times \mathbf{R}^{-S} \subset \mathbf{R}^n$, where superscript C denotes the complement of a set. Let

$$(20) \quad UBP = UBP(\Gamma_\omega) = \bigcap_{S \in \omega; S \neq N} \{[V(S) \setminus \partial V(S)]^C \times \mathbf{R}^{-S}\} \subset \mathbf{R}^n$$

denote payoffs that are unblocked by all $S \in \omega \setminus N$. Then, the old core in (15) can be rewritten as $C_0(\Gamma_\omega) = \partial V(N) \cap UBP(\Gamma_\omega)$, and the concepts of *MNBF* and *EGP* can be defined below.

Definition 4: Given the network game (14), let its *GP* and *UBP* be given in (18) and (20). Let *MNBF* denote its minimum no-blocking frontier and *EGP* its efficient generated-payoffs. Then, *MNBF* and *EGP* are given by

$$(21) \quad MNBF = MNBF(\Gamma_\omega) = \{y \in UBP \mid \exists \text{ no } x \in UBP \text{ such that } x \ll y\}, \text{ and}$$

$$(22) \quad EGP = EGP(\Gamma_\omega) = \partial GP(\Gamma_\omega) = \{y \in GP \mid \exists \text{ no } x \in GP \text{ such that } x \gg y\}.$$

By (21), *MNBF* is the lower boundary or the minimum weakly efficient set of *UBP*. Any payoff vector on (or above) this boundary is unblocked by all $S \in \omega \setminus N$, this is analogous to the TU result that any solution of (4) given in (11) is unblocked by all $S \in \omega \setminus N$. By (22), *EGP* is the upper boundary of *GP*. It will be irrational to choose any $y \in V(N)$ if y is below this boundary; this is analogous to the TU result that it is irrational to split $v(N) < mgp$. Let

$$(23) \quad Z = Z(\Gamma_\omega) = MNBF \cap EGP$$

denote the set of unblocked and efficient generated-payoffs. The next theorem shows that $Z \neq \emptyset$ always holds, which is the NTU counterpart of $mgp = mbp$ implied in Theorem 1.

Theorem 3: Given game (14), let $Z = Z(\Gamma_\omega)$ be given in (23). Then, $Z \neq \emptyset$.

Theorem 3 is proved by a version of Scarf's closed covering theorem (1967a) due to Zhou (1994). Recall that $EGP \subseteq V(N)$ holds in sub-balanced games. By $MNBF \subset UBP$, $Z = MNBF \cap EGP \neq \emptyset$ leads directly to $C_0(\Gamma_\omega) = \partial V(N) \cap UBP \neq \emptyset$ in sub-balanced games.

Now, consider the rationality of choosing a payoff vector from $\partial V(N)$. Similar to the irrationality of splitting $v(N)$ in TU games with $v(N) < mgp$, it is rational to choose $u \in \partial V(N)$ if $GP \subseteq V(N)$ (i.e., if the game is sub-balanced), and will be irrational to choose $u \in \partial V(N)$ if $V(N) \subset GP \setminus \partial GP$ (i.e., if there is $\mathcal{B} \in \mathcal{B}$ and $v \in GP(\mathcal{B})$ such that $v \gg u$). Using our geometric interpretation, it is irrational to choose $u \in \partial V(N)$ if one sees no clouds ($V(N) \subset GP \setminus \partial GP$) and rational to choose $u \in \partial V(N)$ if one sees no peaks ($GP \subset V(N)$).

However, unlike in TU network games in which either $v(N) < mgp$ or $v(N) \geq mgp$ holds, it is possible in NTU network games that neither $V(N) \subset GP \setminus \partial GP$ nor $GP \subset V(N)$ holds, or that one sees both clouds and peaks. The existence of such non-sub-balanced games with $V(N) \not\subset GP \setminus \partial GP$ is what makes the following old NTU core results weaker than the corresponding TU core results in Theorem 1.

Theorem 4: *Given network game (14), let its old core, GP and $MNBF$ be given in (15), (18) and (21) respectively. Then, the following three claims hold:*

- (i) $C_0(\Gamma_\omega) \neq \emptyset$ if $GP \subset V(N)$;
- (ii) $C_0(\Gamma_\omega) = \emptyset$ if $V(N) \subset GP \setminus \partial GP$; and
- (iii) $C_0(\Gamma_\omega) \neq \emptyset \Leftrightarrow$ there exists $x \in \partial V(N)$ and $y \in MNBF$ such that $x \geq y$.

Comparing Theorem 4 with Theorem 1 leads to the following two differences and one similarity between old NTU and TU core results: *i)* sub-balancedness is only a sufficient condition for a non-empty old NTU core, and a necessary and sufficient condition for a non-empty old TU core; *ii)* the irrationality of choosing from $V(N)$ is only a sufficient condition

for an empty old NTU core, whereas the irrationality of splitting $v(N)$ is a necessary and sufficient condition for an empty old TU core; and *iii*) “ $V(N)$ has a payoff vector on or above MNBF” and “ $v(N) \geq mnp$ ” are respectively a necessary and sufficient condition for a non-empty old NTU and TU core.

The NTU counterpart of a TU game’s maximal payoff in (9) is the following concept of efficient payoffs:

Definition 5: *The set of efficient payoffs (EP) for network game (14) is given by*

$$(24) \quad EP = EP(\Gamma_\omega) = \partial(GP \cup V(N)) = \{y \in GP \cup V(N) \mid \exists \text{ no } x \in GP \cup V(N) \text{ with } x \succ y\},$$

where $GP = GP(\Gamma_\omega)$ is the generated payoff given in (18).

Recall that players in a TU network game always split the mp given in (9). Similarly, players in a NTU network game will always choose from the EP given in (24). This answers the question of what subset of payoffs from which players will choose.

Next, consider the question of how to choose a payoff vector from EP . Similar to the stability of splitting mp in the previous section, a payoff vector in EP is stable if no minimal sub-balanced collection \mathcal{B} can block it, because vectors in EP are unblocked by N and deviation by each $S \in \omega \setminus N$ is included in the deviations by B . This leads to the concept of new NTU core defined below. For each $\mathcal{B} \in \mathcal{B}$ with $T = \cup_{S \in \mathcal{B}} S$, let $GP_T(\mathcal{B})$ be its generated payoffs given in (16), and $\partial GP_T(\mathcal{B})$ be the (weakly) efficient set of $\partial GP_T(\mathcal{B})$. Then the core or the new core of (14) can be defined as

$$(25) \quad C = C(\Gamma_\omega) = \{u \in EP(\Gamma_\omega) \mid u_T \notin GP_T(\mathcal{B}) \cap \partial GP_T(\mathcal{B}) \text{ for all } \mathcal{B} \in \mathcal{B}\},$$

where $EP(\Gamma_\omega)$ is given in (24) and \mathcal{B} is given in (5). In words, the new NTU core is the subset of efficient payoffs that are unblocked by all minimal sub-balanced collections.

The above new core is closely related to the following set of minimal sub-balanced

collections that support $Z(\Gamma_\omega)$ in (23):

$$(26) \quad D_0 = D_0(\Gamma_\omega) = \{\mathcal{B} \in \mathcal{B} \mid GP(\mathcal{B}) \in Z(\Gamma_\omega)\}.$$

For each $\mathcal{B} \in D_0$ with its balancing vector w , it will generate the unblocked and efficient generated-payoffs in $GP(\mathcal{B}) \in Z(\Gamma_\omega)$ when each $T \in \mathcal{B}$ is formed for w_T percentage of the time.

As with the TU case, each payoff vector $y \in Z(\Gamma_\omega)$ can possibly be blocked only by the grand coalition N , because the payoff vector y satisfies the rationality for all $S \in \omega \setminus N$ and all $\mathcal{B} \in \mathcal{B}$. Hence, $y \in Z(\Gamma_\omega)$ will be stable if $N \notin \omega$ or if $N \in \omega$ and $y \notin V(N) \setminus \partial V(N)$. It will be useful to consider the stability of each $y \in Z(\Gamma_\omega)$ in the following three cases.

Case 1. $N \notin \omega$, or $N \in \omega$ and $V(N) \subset GP(\Gamma_\omega)$. In this case, it is impossible to have $y \in V(N) \setminus \partial V(N)$, so y is stable against deviations by all $S \in \omega \setminus N$ and all $\mathcal{B} \in \mathcal{B}$.

Case 2. $N \in \omega$ and $GP(\Gamma_\omega) \subset V(N)$. In this case, y is unstable if $y \notin \partial V(N)$ (because it will be blocked by N), and stable if $y \in \partial V(N)$.

Case 3. $N \in \omega$, $V(N) \not\subset GP(\Gamma_\omega)$, and $GP(\Gamma_\omega) \not\subset V(N)$. This case is what makes NTU results different from TU results. The stability of y depends on whether $C_0(\Gamma_\omega) \neq \emptyset$. If $C_0(\Gamma_\omega) = \emptyset$, y is stable because N can not block it (otherwise, $C_0(\Gamma_\omega) \neq \emptyset$ holds); if $C_0(\Gamma_\omega) \neq \emptyset$, the stability of y is similar to Case 2: y is unstable if $y \in V(N) \setminus \partial V(N)$, and stable if $y \notin V(N) \setminus \partial V(N)$. Note that $y \in V(N)$ might not hold in Case 3, but it always holds in Case 2.

The above discussions indicate that the new NTU core in (25) is equal to $Z(\Gamma_\omega)$ in Case 1, $C_0(\Gamma_\omega)$ in Case 2, $C_0(\Gamma_\omega) \cup \{Z(\Gamma_\omega) \cap [V(N) \setminus \partial V(N)]^c\}$ in Case 3 with $C_0(\Gamma_\omega) \neq \emptyset$, and $Z(\Gamma_\omega)$ in Case 3 with $C_0(\Gamma_\omega) = \emptyset$.

Finally, consider the question of what coalitions will be formed. By earlier arguments, rational players will choose from the set of efficient payoffs in (24), so coalitions

formed by rational players shall be either the grand coalition N or the minimal sub-balanced collections in (26) that support those efficient and unblocked payoffs. As with the TU case, the unique balance vector associated with each minimal sub-balanced collection answers the question of how long each of these coalitions will be formed.

The next theorem summarizes the above answers.

Theorem 5: *Given network game (14), let $GP = GP(\Gamma_\omega)$, $Z = Z(\Gamma_\omega)$, $EP = EP(\Gamma_\omega)$, and $C = C(\Gamma_\omega)$ be given in (18) and (23)-(25), respectively. Let $D = D(\Gamma_\omega) \neq \emptyset$ denote the set of efficient and stable coalitions that will be formed. Then, the following three claims hold:*

(i) *rational players will choose from the efficient payoffs in EP ;*

(ii) *the new core is always non-empty and is equal to*

$$(27) \quad C = \begin{cases} C_0 & \text{if } N \in \omega \text{ and } GP \subset V(N); \\ C_0 \cup Z^* & \text{if } N \in \omega; V(N) \not\subset GP; GP \not\subset V(N) \text{ and } C_0 \neq \emptyset; \\ Z & \text{if } N \notin \omega \text{ or if } N \in \omega \text{ and } V(N) \subset GP \text{ or if } N \in \omega; V(N) \not\subset GP; GP \not\subset V(N) \text{ and } C_0 = \emptyset; \end{cases}$$

where $Z^* = Z \cap [V(N) \setminus \partial V(N)]^C$, and $C_0 = C_0(\Gamma_\omega)$ is the old core given in (15);

(iii) *the set of efficient and stable coalitions that will be formed is given by*

$$(28) \quad D = \begin{cases} \{N\} & \text{if } N \in \omega \text{ and } GP \subset V(N); \\ \{N\} \cup D_1 & \text{if } N \in \omega; V(N) \not\subset GP; GP \not\subset V(N) \text{ and } C_0 \neq \emptyset; \\ D_0 & \text{if } N \notin \omega \text{ or if } N \in \omega \text{ and } V(N) \subset GP \text{ or if } N \in \omega; V(N) \not\subset GP; GP \not\subset V(N) \text{ and } C_0 = \emptyset, \end{cases}$$

where $D_1 = D_1(\Gamma_\omega) = \{\mathbb{B} \in D_0 \mid GP(\mathbb{B}) \in Z^*\}$, $D_0 = D_0(\Gamma_\omega)$ and Z^* are given in (26) and (27); and for each $\mathbb{B} \in D$ with its unique balancing vector w , each coalition $T \in \mathbb{B}$ will be formed for w_T unit (or percentage) of the time.

Observe that $C(\Gamma_\omega) \neq \emptyset$ always holds, so there always exists an efficient payoff that is unblocked by any coalition or any sub-balanced collection. Such difference between the new core $C(\Gamma_\omega)$ in (25) and the old core $C_0(\Gamma_\omega)$ in (15) is the consequence of the possible new generated payoffs. In the old core $C_0(\Gamma_\omega)$, players just choose from $V(N)$ and only rule out deviations by each feasible coalition. In the new core $C(\Gamma_\omega)$, players choose from the

game's efficient payoffs (including the possible new and higher payoffs) and rule out not only deviations by each feasible coalition, but also simultaneous deviations by each minimal sub-balanced collection.

6. Conclusion and Discussion

Network games have been predicted as an import and interesting research area in game theory "that are likely to be important for future economic theory" (Allen [2000]). The above analysis has provided a through understanding for the old core in n-person network games, which assigns a non-negative payoff (a set of payoff vectors) for each coalition in a subset of coalitions. The existence of old TU core is fully characterized by the game's sub-balancedness, or by " $mbp \leq v(N)$ ", or by " $mp \leq v(N)$ ". On the other hand, the existence of old NTU core is only partially characterized by sub-balancedness and is fully characterized only by " $V(N)$ has a payoff vector on or above $MNBF$ ".

The paper has explored the possibility that players in a coalition S sometimes could achieve better payoffs than its given payoff $v(S)$ or $V(S)$ by forming a minimal sub-balanced collection whose union is itself. Such explorations led to two important conclusions. First, players will achieve the game's mp (EP) in TU (NTU) network games and they will split mp (choose from EP) within from the always non-empty new core, which coincides with the old core if the old core is non-empty and is equal to the optimal set of mbp (the set of unblocked and efficient generated-payoffs) in TU (NTU) games if the old core is empty. Second, players will form the optimal (efficient) and stable coalitions in minimal sub-balanced collections that support the game's mp (EP) in TU (NTU) games, the unique balancing vector for the minimal sub-balanced collection determines the length (or percentage) of time in which each of the coalitions will be formed.

A striking difference between the above results and those in coalitional games is that coalitions, though not originally defined in the game, could be formed as the union of a minimal sub-balanced collection that generates the game's mp or EP . If such a union is a proper subset of N , it implies that the outsiders have no business in the network game and that the grand coalition will never be formed.

This paper has opened the door for re-examining all existing studies of cooperative game theory in n -person network games. Among such a long list of future studies, readers are encouraged to investigate the properties of the following values and refinements of the new core in an n -person network game: *i*) new Shapley value: replacing $v(N)$ by $mp(\Gamma_\omega)$ in Shapley (1953); *ii*) new nucleolus: replacing $v(N)$ with $mp(\Gamma_\omega)$ in Schmeidler (1969); *iii*) quasi-Shapley value: the vector in $c(\Gamma_\omega)$ that has the shortest distance between $c(\Gamma_\omega)$ and the new Shapley value; *iv*) dual nucleolus: the lexicographical maximizer of the ascending excess vector on $c(\Gamma_\omega)$; and *v*) extensions of (*i-v*) to NTU network games. Keep in mind that the above (*i-v*) are restricted to a given subset of coalitions ω , and that (*ii-iv*) are different selections within the new core.

Appendix

Proof of Theorem 1: The proof consists of three parts.

Part 1. $c_0(\Gamma_\omega) \neq \emptyset \Leftrightarrow (i)$. This is established by applying duality theorem to the linear programming problem (3). Arguments are similar to the proof of the proof of Bondareva-Shapley theorem.

Part 2. $c_0(\Gamma_\omega) \neq \emptyset \Leftrightarrow (ii)$. This is established by studying the linear programming problem (4). Arguments are similar to the proof in Zhao (2001).

Part 3. $c_0(\Gamma_\omega) \neq \emptyset \Leftrightarrow (iii)$. This is established by showing that the maximization

problem (7) and minimization problem (4) are dual to each other. For each $S \neq N$, let $e_S = (x_1, \dots, x_n)^\top \in \mathbf{R}_+^n$ be its incidence vector or the column vector such that $x_i = 1$ if $i \in S$ and $x_i = 0$ if $i \notin S$, and $e = e_N = (1, \dots, 1)^\top$ be a column vector of ones. Then, the dual of the min problem (4) is the following max problem:

$$(29) \quad \text{Max } \{ \sum_{S \in \omega \setminus N} w_S v(S) \mid w_S \geq 0, S \in \omega \setminus N; \text{ and } \sum_{S \in \omega \setminus N} w_S e_S \leq e \}.$$

Our purpose is to show that (29) is equivalent to the max problem (7). This is achieved in the following five steps.

Step 1. We study a new max problem that is equivalent to (29) by adding new variables y_S for all $S \notin \omega$ and $S \neq N$. This new max problem is given as below:

$$(30) \quad \text{Max } \{ \sum_{S \neq N} y_S u(S) \mid y_S \geq 0, S \neq N; \text{ and } \sum_{S \neq N} y_S e_S \leq e \},$$

where $u(S) = v(S)$ for all $S \in \omega \setminus N$, and $u(S) = 0$ for $S \notin \omega$ and $S \neq N$. Note that there are $m' = |\omega \setminus N|$ variables in (29) and $m = (2^n - 2)$ variables in (30). It is straightforward to see the equivalence between (29) and (30), because they have the same maximal value and for each optimal solution y of (30), $w_{\omega \setminus N} = \{y_S \mid S \in \omega\}$ is an optimal solution of (29).

Step 2. We show that the inequality constraints in (30) can be replaced by equation constraints. Let $Ay \leq e$ and $y \geq 0$ denote the constraints in (30), where $A = [e_S \mid S \neq N] = A_{n \times m}$ is the constraint matrix, $m = (2^n - 2)$, and $y \in \mathbf{R}_+^m$ is the choice variable. Let the rows of A be a_1, \dots, a_n , and for each feasible y , let $T = T(y) = \{i \mid a_i y < 1\}$ be the set of loose constraints, so $N \setminus T = \{i \mid a_i y = 1\}$ is the set of binding constraints.

If $T(y) \neq \emptyset$, let z be defined as: $z_S = y_S + (1 - a_i y)$ if $S = \{i\}$, for each $i \in T$, and $z_S = y_S$ if $S \neq \{i\}$ for all $i \in T$. One sees that $z > y$ (i.e., $z \geq y$ and $z \neq y$) and $T(z) = \emptyset$. Hence, for any y with $T(y) \neq \emptyset$, there exists $z \geq 0$, $Az = e$ such that $\sum_{S \neq N} y_S u(S) \leq \sum_{S \neq N} z_S u(S)$. This shows that

the feasible set of (30) can be replaced by $\{z \mid z \geq 0, Az = e\}$, without affecting the maximum value. So the maximization problem in (30) is equivalent to the following problem:

$$(31) \quad \text{Max } \{\sum_{S \neq N} y_S u(S) \mid Ay = e, \text{ and } y \geq 0\}.$$

Step 3. Note that for each feasible y in (31), $\mathcal{B}(y) = \{S \mid y_S > 0\}$ is a balanced collection. We now establish the one-to-one relationship between the extreme points of (31) and the minimal balanced collections. Let y be an extreme point of (31), assume by way of contradiction that $\mathcal{B}(y)$ is not minimal. Then, there exists a balanced sub-collection $B' \subset \mathcal{B}(y)$ with balancing vector z . By the definition of a balancing vector, $z_S > 0$ implies $y_S > 0$. Choose $t > 0$ and $t \leq \text{Min } \{y_S / |z_S - y_S| \mid \text{all } S \text{ with } y_S \neq z_S\}$, one has

$$w = y - t(y-z) \geq 0, w' = y + t(y-z) \geq 0.$$

$Ay = e$ and $Az = e$ lead to $Aw = e$ and $Aw' = e$. But $y = (w+w')/2$ and $w \neq w'$ contradict the assumption that y is an extreme point. So $\mathcal{B}(y)$ must be minimal.

Next, let $\mathcal{B} = \{T_1, \dots, T_k\}$ be a minimal balanced collection with a balancing vector z , and assume by way of contradiction that z is not an extreme point of (31), so there exists $w \neq w'$ such that $z = (1-\lambda)w + \lambda w'$ for some $0 < \lambda < 1$. By $w \geq 0$ and $w' \geq 0$, one has

$$\{S \mid w_S > 0\} \subseteq \mathcal{B} = \{S \mid z_S > 0\}, \text{ and } \{S \mid w'_S > 0\} \subseteq \mathcal{B} = \{S \mid z_S > 0\}.$$

The above two expressions show that both w and w' are balancing vectors for some sub-collections of \mathcal{B} . Because \mathcal{B} is minimal, one must have $w = w' = z$, which contradicts $w \neq w'$. Therefore, z must be an extreme point of (31).

Step 4. By the standard results in linear programming, the maximal value of (31) is achieved among the set of its extreme points, which are equivalent to the set of the minimal balanced collections, so (31) is equivalent to the max problem given below:

$$(32) \quad \text{Max } \{gp(\mathcal{B}) \mid N \notin \mathcal{B}, \text{ and } \mathcal{B} \text{ is a minimal balanced collection of } \mathcal{N}\}.$$

Step 5. Finally, we show that (32) is equivalent to (29). Let the maximal value of (32) be $gp(\mathcal{B}')$, where \mathcal{B}' is a maximal collection. Consider $\mathcal{B} = \mathcal{B}' \cap \omega$. Since \mathcal{B}' is a minimal sub-balanced collection of \mathcal{N} , \mathcal{B} is a minimal sub-balanced collection of ω . It follows from the definitions of (29)-(32) that $gp(\mathcal{B}) = gp(\mathcal{B}')$ is the maximal value of (29). This completes the proof of Part III.

This shows that (29) is equivalent to the maximization problem (7) for mgp , which completes part 3 of the proof. **Q.E.D**

Proof of Theorem 2: The discussion between Definition 2 and the theorem serves as a proof of the theorem. **Q.E.D**

Our proof for Theorem 3 uses the following lemma on open covering of the simplex $\Delta^N = X(I) = \{x \in \mathbf{R}_+^n \mid \sum_{i \in N} x_i = I\}$.

Lemma 1 (Scarf [1967a], Zhou [1994]): *Let $\{C_S\}$, $S \neq N$, be a family of open subsets of Δ^N that satisfy $\Delta^{N \setminus \{i\}} = \{x \in \Delta^N \mid x_i = 0\} \subset C_{\{i\}}$ for all $i \in N$, and $\cup_{S \neq N} C_S = \Delta^N$, then there exists a balanced collection of coalitions \mathcal{B} such that $\cap_{S \in \mathcal{B}} C_S \neq \emptyset$.*

Proof of Theorem 3: Let UBP be the set of unblocked payoffs in (20), and EGP be the boundary or (weakly) efficient set of the generated payoff in (22). We shall first show that $UBP \cap EGP \neq \emptyset$.

For each coalition $S \neq N$, let $W_S = \{Int V(S) \times \mathbf{R}^{-S}\} \cap EGP$ be an open (relatively in EGP) subset of EGP , where $Int V(S) = V(S) \setminus \partial V(S)$ is the interior of $V(S)$. For each minimal balanced collection of coalitions \mathcal{B} , we claim that

$$(33) \quad \bigcap_{S \in \mathcal{B}} W_S = \emptyset$$

holds. If (33) is false, there exists $y \in EGP$ and $y \in \text{Int } V(S) \times \mathbf{R}^{-S}$ for each $S \in \mathcal{B}$. We can now find a small $t > 0$ such that $y + te \in \text{Int } V(S) \times \mathbf{R}^{-S}$ for each $S \in \mathcal{B}$, where e is the vector of ones. By the definition of (16) and (18), $y + te \in GP(\mathcal{B}) = \bigcap_{S \in \mathcal{B}} \{V(S) \times \mathbf{R}^{-S}\} \subset GP$, which contradicts $y \in EGP$. This proves (33).

Now, suppose by way of contradiction that $UBP \cap EGP = \emptyset$. Then, $EGP \subset UBPC^C$, where superscript C denotes the complement of a set. The definition of W_S and

$$UBPC^C = \{\bigcap_{S \neq N} \{[V(S) \setminus \partial V(S)]^C \times \mathbf{R}^{-S}\}\}^C = \bigcup_{S \neq N} \{\text{Int } V(S) \times \mathbf{R}^{-S}\}$$

together lead to $\bigcup_{S \neq N} W_S = EGP$, so $\{W_S\}, S \neq N$, is an open cover of EGP .

Because the set of generated payoffs is comprehensive and bounded from above, and the origin is in its interior (by $\partial V(i) > 0$, all i), the following mapping from EGP to Δ^N :

$$f: x \rightarrow x / \sum x_i,$$

is a homeomorphism. Define $C_S = f(W_S)$ for all $S \subseteq N$, one sees that $\{C_S\}, S \neq N$, is an open cover of $\Delta^N = f(EGP)$.

For each $i \in N$, $\partial V(i) > 0$ leads to $EGP \cap \{x \in \mathbf{R}^n \mid x_i = 0\} \subset W_{\{i\}}$, which in turn leads to $\Delta^{N \setminus \{i\}} = \{x \in \Delta^N \mid x_i = 0\} = f(EGP \cap \{x \in \mathbf{R}^n \mid x_i = 0\}) \subset C_{\{i\}} = f(W_{\{i\}})$. Therefore, $\{C_S\}, S \neq N$, is an open cover of Δ^N satisfying the conditions of Scarf-Zhou open covering theorem, so there exists a balanced collection of coalitions \mathcal{B}_0 such that $\bigcap_{S \in \mathcal{B}_0} C_S \neq \emptyset$, or that

$$(34) \quad \bigcap_{S \in \mathcal{B}_0} W_S \neq \emptyset,$$

which contradicts (33). Hence, $UBP \cap EGP \neq \emptyset$.

For each $x \in UBPC^C$, we claim $x \in MNBF$. If this is false, we can find a small $\tau > 0$ such that $x - \tau e \in UBPC^C$. Let $\mathcal{B} \in \mathcal{B}$ be the minimal balanced collection of coalitions such

that $x \in GP(\mathcal{B}) = \bigcap_{S \in \mathcal{B}} \{V(S) \times \mathbf{R}^{-S}\}$. Then, $x - \tau e \in \text{Int } V(S) \times \mathbf{R}^{-S}$ for each $S \in \mathcal{B}$, which contradicts $x - \tau e \in UBP$. Therefore, $MNBF \cap EGP = UBP \cap EGP \neq \emptyset$. **Q.E.D**

Proof of Theorem 4: It follow from the discussions preceding the theorem. **Q.E.D**

Proof of Theorem 5: The conclusions follow from the discussions between Definition 5 and the theorem. **Q.E.D**

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