HANDBOOK OF
Game Theory and
Industrial Organization
VOLUME I
Theory

Edited by
Luis C. Corchón • Marco A. Marini
The first volume of this wide-ranging Handbook contains original contributions by world-class specialists. It provides up-to-date surveys of the main game-theoretic tools commonly used to model industrial organization topics. The Handbook covers numerous subjects in detail including, among others, the tools of lattice programming, supermodular and aggregative games, monopolistic competition, horizontal and vertically differentiated good models, dynamic and Stackelberg games, entry games, evolutionary games with adaptive players, asymmetric information, moral hazard, learning and information sharing models.

'Game theoretic methods are central in the study of oligopoly markets. The surveys in this Handbook provide a broad introduction to the relevant game theory topics and their applications in oligopoly theory. There is an emphasis on recent developments, such as lattice theory and supermodular games. The Handbook will be a valuable resource for researchers and students.'
– Robert Porter, Northwestern University, US
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Contents

1 Introduction
   Luis C. Corchón and Marco A. Marini

PART I  BASIC GAMES IN INDUSTRIAL ORGANIZATION

2 Strategic complementarities in oligopoly
   Xavier Vives

3 On the Cournot and Bertrand oligoplies and the theory
   of supermodular games
   Rabah Amir

4 Aggregative games
   Martin Kaare Jensen

5 Monopolistic competition without apology
   Jacques-François Thisse and Philip Uschev

6 Oligopoly and product differentiation
   Jean J. Gabszewicz and Ornella Tarola

7 Oligopolistic competition and welfare
   Robert A. Ritz

PART II  DYNAMIC GAMES IN INDUSTRIAL ORGANIZATION

8 Dynamic games
   Klaus Ritzberger

9 Strategic refinements
   Carlos Pimienta

10 Stackelberg games
    Ludovic A. Julien

11 Entry games and free entry equilibria
    Michele Polo

12 Evolutionary oligopoly games with heterogeneous adaptive players
    Gian Italo Bischi, Fabio Lampantia and Davide Radi

PART III  GAMES OF COLLUSION IN INDUSTRIAL ORGANIZATION

13 Coalitions and networks in oligopolies
    Francis Bloch

14 TU oligopoly games and industrial cooperation
    Jingang Zhao

PART IV  INFORMATION GAMES

15 Trading under asymmetric information: Positive and normative
   implications
   Andrea Attar and Claude d’Aspremont

16 Moral hazard: Base models and two extensions
   Inès Macho-Stadler and David Pérez-Castrillo

17 Learning in markets
   Amparo Urbano

18 Information sharing in oligopoly
   Sergio Currafini and Francesco Feri

Index

537
TU Oligopoly Games and Industrial Cooperation

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Abstract

This chapter surveys existing results and lists nine future areas in TU oligopoly games, which are both theoretically interesting and empirically important. On the theory side, they make advances on the refinements and applications of the core, one of the most important solutions in cooperative game theory. On the empirical side, TU oligopoly games allow one to model and analyze industrial cooperation and help understand the forces behind industrial changes as well as the effects of regulatory policies.

Contents

1 Introduction ........................................... 2
2 The core as a solution in three game models ................. 3
  2.1 The core in coalitional games ...................... 6
  2.2 The $\alpha$-core and $\beta$-core in normal form TU games .... 7
  2.3 The $\gamma$-core, $\delta$-core and their variations in partition function games ... 9
3 Core and its refinements as candidates of monopoly solutions .......... 12
  3.1 The equilibrium expressions in oligopoly models .... 13
  3.2 The core in oligopoly TU games .................. 19
     3.2.1 Non-empty core as a precondition for horizontal mergers .... 20
     3.2.2 The core with weak synergy .................. 22
     3.2.3 The convexity in oligopoly games .............. 24
  3.3 Refinements of the core .......................... 27
  3.4 Extensions ................................... 35
4 Stable partitions as candidates of non-monopoly solutions .......... 38
5 Empirical studies of the core ................................ 42
6 Conclusion and future study ................................ 44

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1 Introduction

This chapter surveys existing results and lists nine future areas in TU oligopoly games and industrial cooperation or precisely cooperative oligopoly games with transferable utilities (TU). Such results and future research are both empirically important and theoretically interesting.

On the empirical side, the model of TU oligopoly games is the proper tool to study industrial cooperation, ranging from early divisions of labor to modern merger contracts; its applications help understand the structural changes in industries and the effects of regulatory policies.\(^1\) For example, empty-core theory has provided an understanding about the U.S. consolidation movement of the late nineteenth century, which actually originated the field of Industrial Organization (McWilliams and Keith 1994).

Core theory allows one to estimate the merging costs or the transaction costs of horizontal mergers (Zhao 2009a). Reductions in merging costs provide a new explanation for the two greatest merger waves around the turns of the twentieth and the twenty-first centuries (Zhao 2009b). Similar cost reductions by Solvency II (enacted on January 1, 2016) likely will drive more mergers and acquisitions in the EU insurance industry (Stoyanova and Gruendl 2014).

On the theory side, the results are advances in the refinements and applications of the core, which is the most important solution in cooperative game theory. They are developed around the stability of a monopoly merger contract. It first converts the oligopoly to a TU coalitional game or a partition function game and then characterizes the core. The main task is to identify conditions on the parameters in an oligopoly for a non-empty core. One sufficient condition for a non-empty core is convexity or

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\(^1\) See Daughety and Reinganum (2017) for survey.
supermodularity, whose existence is known only in some linear oligopolies.

It goes without emphasizing the importance of core theory in industrial organization, because non-empty core and profitability are the two preconditions for each horizontal merger. It should be pointed out that NTU oligopoly games or cooperative oligopoly games with non-transferable utilities (NTU) are not surveyed here. The model of NTU games is the tool to study collusion such as illegal cartel agreements,\(^2\) which are not in the mainstream of industrial organization.

The rest of this survey is organized as follows: Section 2 reviews three game models and defines their core solutions and some refinements that are relevant in oligopolies. Section 3 first reviews ten oligopoly models and their equilibrium expressions, it then reviews the existence and refinements of the core, and finally it lists seven extensions. Section 4 reviews the results on non-monopoly partitions. Section 5 reviews empirical studies of the core. Section 6 concludes with a brief discussion about future research.

## 2 The core as a solution in three game models

This section reviews the core as a cooperative solution in three forms of games or three models: coalitional TU games or simply coalitional games (also called games in characteristic form or games in coalitional form, von Neumann and Morgenstern 1944), normal form games (also called strategic games, Nash 1950), and partition function games (Thrall and Lucas 1963). These games are defined below.

Let \( N = \{1,\ldots,n\} \) be the set of players or firms. Each subset \( S \subseteq N \) is called an alliance or a coalition or a merger. Each partition \( \Delta = \{S_1, S_2, \ldots, S_h\} \) of \( N \) is called a coalition structure or market structure, representing a set of \( h \) simultaneous mergers in which each merger \( S_j \) has \( k_j = |S_j| \) members (so \( \sum_{j=1}^{h} k_j = n \)).\(^3\)


\(^3\) A partition \( \Delta = \{S_1, S_2, \ldots, S_h\} \) satisfies: \( S_j \neq \emptyset, \cup S_j = N, \) and \( S_i \cap S_j = \emptyset, \) all \( i \neq j. \)
A coalitional game (von Neumann and Morgenstern 1944) is a set function given by
\[ \Gamma_c = \{N, v(\cdot)\}, \tag{1} \]
specifying a nonnegative joint payoff or profit \( v(S) \) for each coalition \( S \subseteq N \). The central question here is how to split the grand coalition’s payoff \( v(N) \) among the \( n \) players; this implicitly assumes that the grand coalition’s payoff (such as monopoly profit) is optimal or maximal among all coalitions and all partitions of \( N \).

A normal form game (Nash 1950) is given by
\[ \Gamma = \{N, X_i, u_i\}, \tag{2} \]
specifying a choice set \( X_i \subseteq R^{k_i} \) in \( k_i \)-dimensional Euclidian space and a payoff function \( u_i(x) \), \( x = (x_1, \ldots, x_n) \in X = \Pi_{j=1}^n X_j \), for each player \( i \in N \). Game (2) is called a normal form TU game if all \( u_i \) are transferable, such as dollars. The central question here is what is the solution or a list of choices \( x = (x_1, \ldots, x_n) \) that rational players will choose.

A partition function game (Thrall and Lucas 1963) is given by
\[ \Gamma_p = \{N, \phi(\cdot)\}, \tag{3} \]
specifying a vector of joint payoffs \( \phi(\Delta) = \{\phi_S = \phi_S(\Delta) \mid S \in \Delta\} \) for each partition \( \Delta \) and each of its coalitions \( S \in \Delta \). One of the central questions here is the same as that in a coalitional game: how to split the grand coalition’s payoff \( v(N) = \phi_N \), assuming that \( v(N) \) is the maximum among all partitions.

The following four assumptions (Zhao 2016) are implicitly assumed to support a variety of solutions for the games in (1-3):

This also represents a \( h \)-firm \( n \)-product multi-product oligopoly, in which each firm \( S_j \) (or simply \( j \)) produces \( k_j = |S_j| \) products. As shown in Zhao (2012), the equilibrium expressions for such multi-product oligopoly are identical to that for the postmerger equilibria. See Fauli-Oller (2017) for survey on other studies on mergers.
A1 (Assumption 1) Players are able to take collective actions.

A2 Players are unable to take any form of coordinated or collective actions.

A3 Players are able to costlessly negotiate and enforce a joint action.

A4 Given a partition $\Delta = \{S_1, S_2, \ldots, S_h\}$, A3 holds for each $S \in \Delta$, and A2 holds for each $T \notin \Delta$ such that there are $i \neq j$, $T \cap S_i \neq \emptyset$ and $T \cap S_j \neq \emptyset$.

A1 applies in most situations, A2 characterizes the original Prisoner’s Dilemma game in which the two players have no access to any form of communication or coordination or agreements. If they could coordinate their choices or make deals by using a joint counsel, the nature of the game will become that under A1 or A3 and thus invalidate the Nash equilibrium predicted by A2.

Note also that players under A4 can negotiate and enforce a joint action if they belong to the same coalition, but they can not take collective or coordinated actions if they are from two or more different coalitions. Thus, A2 is a special case of A4 for the finest partition or premerger structure $\Delta_0 = \{\{1\}, \ldots, \{n\}\}$, and A3 is another special case for the coarsest partition or monopoly structure $\Delta_m = \{N\}$.

Because these assumptions determine or limit a player’s rationality, they are the foundations of game theory upon which various solutions or theories are built. For example, A2, A3 and A4 are the foundations of noncooperative solution or Nash equilibrium (Nash 1950), cooperative solutions (Shapley 1955, von Neumann and Morgenstern 1944), and hybrid solutions (Zhao 1992), respectively.

Care needs to be taken when applying these assumptions in a particular game. For example, the actions for a coalition $S$ under both A1 and A2 in a normal form game (2) are the vectors of their choices given by $x_S = \{x_j | j \in S\} \in X_S = \Pi_{j \in S} X_j$; but their actions in a coalitional game (1) are the splits of their joint payoff given by $\theta_S = \{\theta_j | j \in S\}$, which satisfies $\Pi_{j \in S} \theta_j = v(S)$, and $\theta_j \geq 0$, all $j$.  

5
2.1 The core in coalitional games

Given a coalitional game (1), a split of \( v(N) \) is a payoff vector \( \theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}_n^+ \) such that \( \Sigma_{j=1}^n \theta_j = v(N) \), with \( \theta_j \) as player \( j \)'s payoff, all \( j \). A split \( \theta \) is rational for a coalition \( S \subseteq N \) (or undominated or unblocked by \( S \)) if \( \Sigma_{j \in S} \theta_j \geq v(S) \), and \( \theta \) is in the core (or a core vector) if it is rational for all \( S \subseteq N \). This was first defined by Shapley (1955) as given here:

**Definition 1 (Shapley 1955)** The core of game (1) is the set of the splits of \( v(N) \) that are rational for all proper coalitions. Precisely, this is given by

\[
\text{Core}(\Gamma_c) = \{ \theta \in \mathbb{R}_n^+ | \Sigma_{j=1}^n \theta_j = v(N), \text{ and } \Sigma_{j \in S} \theta_j \geq v(S), \text{ all } S \neq N \}. \tag{4}
\]

Lemma 1 below summarizes two complete arguments for a non-empty core.

**Lemma 1** Given (4), the following three arguments are equivalent: i) \( \text{Core}(\Gamma_c) \neq \emptyset \); ii) the game is balanced (Bondareva 1962, Shapley 1967); and iii) the grand coalition’s payoff is above the minimum no-blocking payoff (Zhao 2001b).

Specifically, argument (ii) holds if \( \Sigma_{T \in B} w_T v(T) \leq v(N) \) holds for each balanced collection of coalitions \( B = \{T_1, ..., T_k \} \) with a balancing vector \( w = \{w_T | T \in B\} \); and argument (iii) holds if \( v(N) \geq mnbp \) holds, where \( mnbp \) is the game’s minimum no-blocking payoff given by

\[
mnbp = \text{Min} \{ \Sigma_{j \in N} \theta_j | \theta \in R_n^+, \text{ and } \Sigma_{j \in S} \theta_j \geq v(S), \text{ all } S \neq N \}. \tag{5}
\]

The above \( mnbp \) method for core existence has an intuitive interpretation and it enables one to estimate the transaction costs of horizontal mergers (Zhao 2009a).

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4 Shapley first coined the term core solution during 1952–1953 in one of his conversations with Shubik, and Gillies first used the term core during the same period, referring to some intersections of the stable sets. Gillies (1959) had been mistakenly cited in most previous studies as the first paper that defined the core. See Zhao (2016) for the history of the core.

5 A collection of coalitions \( B = \{T_1, ..., T_k \} \) is balanced if it has a balancing vector \( w \), or a positive weight \( w_T > 0 \) for each \( T \in B \), such that for each player \( i \in N \), \( \Sigma_{T \in B(i)} w_T = 1 \) holds, where \( B(i) = \{ T \in B | i \in T \} \) is the subcollection of coalitions to which player \( i \) belongs.
2.2 The \(\alpha\)-core and \(\beta\)-core in normal form TU games

Given a coalition \(S\) in the normal form TU game (2), recall that its choice vector is given by \(x_S = \{x_j|j \in S\} \in X_S = \Pi_{j \in S} X_j\). Let \(x_{-S} = \{x_j|j \notin S\} \in X_{-S} = \Pi_{j \notin S} X_j\) be the outsiders’ choice vector, and rearrange \(x = (x_1, ..., x_n) \in X = \Pi_{j=1}^n X_j\) as \(x = (x_S, x_{-S})\), so \(u_i(x) = u_i(x_S, x_{-S})\) for all \(i\). Then, the coalition’s joint payoffs under the \(\alpha\)- and \(\beta\)-beliefs are defined by

\[
v_{\alpha}(S) = \max_{x_S \in X_S} \min_{x_{-S} \in X_{-S}} \left\{ \sum_{j \in S} u_j(x_S, x_{-S}) | x_S \in X_S \right\}
\]

and

\[
v_{\beta}(S) = \min_{x_{-S} \in X_{-S}} \max_{x_S \in X_S} \left\{ \sum_{j \in S} u_j(x_S, x_{-S}) | x_S \in X_S \right\},
\]

respectively. \(v_{\alpha}(S)\) is often called the guaranteed or worst payoff, because \(S\) can guarantee a joint payoff no less than \(v_{\alpha}(S)\) by choosing some \(\pi_S\) (i.e., \(\sum_{j \in S} u_j(\pi_S, x_{-S}) \geq v_{\alpha}(S)\) for all \(x_{-S}\)). On the other hand, \(S\) can not be prevented from receiving at least \(v_{\beta}(S)\), as they have a best response function

\[
x_{S}^* = x_S(x_{-S}) = \arg\max_{x_S \in X_S} \sum_{j \in S} u_j(x_S, x_{-S}) | x_S \in X_S
\]

such that \(\sum_{j \in S} u_j(x_S(x_{-S}), x_{-S}) \geq v_{\beta}(S)\) for each \(x_{-S}\).

By \(\max\{\sum_{j \in N} u_j(x)|x \in X\} = v(N) = v_{\alpha}(N) = v_{\beta}(N)\), the grand coalition’s payoff is the same under both beliefs. An updated version of the \(\alpha\)-and \(\beta\)-cores in Aumann (1959) are given here:

**Definition 2 (Aumann 1959)** Given a normal form TU game (2) and its coalitional payoffs \(v_{\alpha}(S)\) and \(v_{\beta}(S)\) in (6-7), its \(\alpha\)- and \(\beta\)-coalitional games are

\[
\Gamma_{\alpha} = \{N, v_{\alpha}(\cdot)\} \text{ and } \Gamma_{\beta} = \{N, v_{\beta}(\cdot)\}, \text{ and}
\]

the cores of above \(\Gamma_{\alpha}\) and \(\Gamma_{\beta}\) are called the \(\alpha\)-core and the \(\beta\)-core, respectively.

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\(^6\) ArgMax denotes the set of maximal solutions for each maximization problem; precisely, given \(\max\{f(x)|x \in X\}\), one has \(\text{ArgMax}\{f(x)|x \in X\} = \{y \in X|f(y) \geq f(x), \text{ all } x \in X\}\).
As shown in Zhao (1999a, 156), an empty $\alpha$-core means that for each $\theta$ satisfying $\theta \geq 0$ and $\sum_{j \in S} \theta_j = v(N)$, there exists $S$ and $x_S \in X_S$ such that $\sum_{j \in S} u_j(x_S, x_{-S}) > \sum_{j \in S} \theta_j$ for all $x_{-S}$, and an empty $\beta$-core means the existence of $S$ with a reaction function $x_S^* = x_S(x_{-S})$ in (8) such that $\sum_{j \in S} u_j(x_S(x_{-S}), x_{-S}) > \sum_{j \in S} \theta_j$ for all $x_{-S}$.

Thus, an empty $\alpha$-core implies an empty $\beta$-core, so a non-empty $\beta$-core implies a non-empty $\alpha$-core, or $\text{Core}(\Gamma_\beta) \subseteq \text{Core}(\Gamma_\alpha)$ holds. This can also be understood by the following interpretation due to Jianbo Zhang of University of Kansas (Zhang 2016).

For the $\alpha$-core, imagine that the outsiders have a spy in $S$ and thus know each action taken by $S$. Consequently, all actions taken by $S$ are doomed to be disastrous, and the best $S$ could do is damage control or choose the best of the worst given by $v_\alpha(S)$. On the other hand, one imagines, for the $\beta$-core, that $S$ have a spy in $N \setminus T$ and know each action taken by the outsiders. In this case, each of outsiders’ actions will lead to the best outcome for $S$, and the worst harm that the outsiders could do to $S$ is given by $v_\beta(S)$. Having a spy is better than being spied on, so one has $v_\beta(S) \geq v_\alpha(S)$ and thus $\text{Core}(\Gamma_\beta) \subseteq \text{Core}(\Gamma_\alpha)$.

The general existence of NTU $\alpha$-core was established by Scarf (1971). He showed that the normal form game (2) has a non-empty NTU $\alpha$-core if (i) all choice sets are compact and convex, and (ii) all payoff functions are continuous and quasi-concave. This has been extended to a non-empty TU $\alpha$-core by adding the assumption of weak separability (Zhao 1999c), and a non-empty TU $\beta$-core by adding the assumption of strong separability (Zhao 1999a). These two extensions$^7$ are relevant in oligopoly models, which are summarized here.

**Lemma 2** Let $C_\alpha = \text{Core}(\Gamma_\alpha)$ and $C_\beta = \text{Core}(\Gamma_\beta)$ be the TU $\alpha$- and $\beta$-cores in (2). Then, (i) $C_\alpha \neq \emptyset$ if 1) all $X_i$ are compact and convex, 2) all $u_i(x)$ are

continuous and quasi-concave, and 3) weak separability holds (Zhao 1999c); ii) $C_\beta \neq \emptyset$ if 1) all $X_i$ are compact and convex, 2) all $u_i(x)$ are continuous and quasi-concave, and 3) strong separability holds (Zhao 1999a).

Roughly speaking, strong (weak) separability requires that the outsiders’ choices that minimize the insiders’ joint payoff in (7) (in (6)) also minimize each insider’s individual payoff in a relevant range (at a relevant point). The precise statements of these two conditions are not reviewed here because they both automatically hold in oligopoly models. Readers are referred to Zhao (1999a), Zhao (1999c), and Meinhardt (2002, 69-88) for details and numerical examples.

2.3 The $\gamma$-core, $\delta$-core and their variations in partition function games

Given a partition $\Delta = \{S_1, S_2, ..., S_h\}$ in the partition function game (3), consider the deviation by, or formation of, a new coalition $S = \{i_1, ..., i_k\} \notin \Delta$. Let the set of those partitions of which $S$ is a member be denoted by

$$\Pi(S) = \{\Delta' \in \Pi | \Delta' = \{S, T_1, ..., T_m\}\}$$

where $\Pi$ is the set of all partitions of $N$. Before deviating, the insiders or players in $S$ are assumed to have hypothesized or believed a reasonable reaction to their deviation by the outsiders in $N \backslash S = \{j | j \notin S\}$.

Six possible reactions based on six kinds of beliefs are known and are reviewed here. These beliefs lead to six core solutions for the game (3): the $\gamma$-core and $\delta$-core in Hart and Kurz (1983), $\alpha^*$-core in Zhao (1996, 2013), $e$-core in Yong (2004), $j$-core in Lekeas (2013), and $f$-core in Lekeas and Stamatopoulos (2014). Note that this list excludes those core-refinements (such as the $lf$-core of Currafini and Marini (2003).
reviewed at end of subsection 3.3) in a normal form game that are not defined for
partition function games.\footnote{It also excludes related studies such as the core in partition function games from a common pool resource (Funaki and Yamato 1999) and the axiomatization of such cores in partition function games (Bloch and van den Noweland 2014).}

For simplicity, all definitions here focus on the coarsest or monopoly partition
$\Delta_m = \{N\}$, which is extended to a general or non-monopoly partition in section 4.

1) The breakup belief or $\gamma$-belief (Hart and Kurz 1983): insiders believe that
the $(n - k)$ outsiders in $N \setminus S$ will breakup into singletons or the new partition is
$\Delta_\gamma = \Delta_\gamma(S, \Delta_m) = \{S, \{j_1\}, \ldots, \{j_{n-k}\}\} \in \Pi(S)$, so the insiders’ payoff and the
$\gamma$-coalitional game are

$$v_\gamma(S) = \phi_S(\Delta_\gamma), \text{ all } S; \text{ and } \Gamma_\gamma = \{N, v_\gamma(\cdot)\}. \quad (11)$$

2) The loyal belief or $\delta$-belief (Hart and Kurz 1983): insiders believe that outsiders
are loyal to each other and stay in the coalition $N \setminus S$, so the new partition is $\Delta_\delta = \Delta_\delta(S, \Delta_m) = \{S, N \setminus S\} \in \Pi(S)$, and their payoff and the $\delta$-coalitional game are

$$v_\delta(S) = \phi_S(\Delta_\delta), \text{ all } S; \text{ and } \Gamma_\delta = \{N, v_\delta(\cdot)\}. \quad (12)$$

3) The cautious belief or $\alpha^*$-belief (Zhao 1996, 2013): insiders are cautious about
their smallest payoff at the worst partition: $\Delta_{\alpha^*} \equiv \Delta_{\alpha^*}(S) = \{S, T_1^{\alpha^*}, \ldots, T_m^{\alpha^*}\} \in \Pi(S)$, or they believe that the outsiders partition themselves to minimize the insiders’
joint payoff, so the insiders’ payoff and the $\alpha^*$-coalitional game are

$$v_{\alpha^*}(S) = \phi_S(\Delta_{\alpha^*}), \text{ all } S; \text{ and } \Gamma_{\alpha^*} = \{N, v_{\alpha^*}(\cdot)\}, \quad (13)$$

where $\Delta_{\alpha^*} = \Delta_{\alpha^*}(S)$ is the solution of $Min\{\phi_S = \phi_S(\Delta')|\Delta' \in \Pi(S)\}$. Note that this
cautious or worst partition $\Delta_{\alpha^*}$ is independent of all current partitions $\Delta$.

4) The efficient belief or $e$-belief (Yong 2004): insiders believe that the outsiders
choose an efficient partition (or optimal partition in TU games) for themselves among
all partitions of $N\setminus S$, so the insiders’ payoff and the $e$-coalitional game are

$$v_e(S) = \phi_S(\Delta_e), \text{ all } S; \text{ and } \Gamma_e = \{N, v_e(\cdot)\},$$

(14)

where $\Delta_e = \Delta_e(S) = \{S, T^e_1, ..., T^e_{m(e)}\}$ solves $\max\{\sum_{T \in \Delta\setminus S} \phi_T(\Delta') | \Delta' \in \Pi(S)\}$. This efficient partition $\Delta_e$ is also independent of all current partitions $\Delta$.

The next two beliefs assume that the payoffs for each $\Delta = \{S_1, S_2, ..., S_h\}$ are determined by the number and sizes of its coalitions or precisely by $h$ and $s_i = |S_i|$, $i = 1, ..., h$. Such property holds when players are symmetric within each coalition.

5) $j$-belief (Lekeas 2013): let $s = |S|$ be the cardinality or the number of insiders for each $S \neq N$, then a $j$-belief is an integer-to-integer function $j(s), 1 \leq j(s) \leq n - s$, for $s = 1, ..., n - 1$, defining the belief for all coalitions with $s$ members that outsiders are divided into $j(s)$ coalitions and the worst of such $j$-partitions will be formed, so the insiders’ payoff and the $j$-coalitional game are

$$v_j(S) = \phi_S(\Delta^*_j(s)), \text{ all } S; \text{ and } \Gamma_j = \{N, v_j(\cdot)\},$$

(15)

where $\Delta^*_j(s) = \{S, T^*_1, ..., T^*_j(s)\}$ solves $\min\{\phi_S = \phi_S(\Delta^*_j(s)) | \Delta^*_j(s) \in \Delta_j(S)\}$, with $\Delta_j(S) = \{\Delta | \Delta = \{S, T_1, ..., T_j(s)\} \in \Pi(S)\}$ as the set of all $j$-partitions or all partitions in which the outsiders are divided into $j(s)$ coalitions.\(^9\)

The next belief further assumes that the payoff $\phi_S(\Delta)$ of each $S \in \Delta = \{S_1, S_2, ..., S_h\}$ is determined by $h$ or the number of coalitions in $\Delta$. This property holds in standard symmetric homogeneous Cournot model with linear cost.

6) The probability belief or $f$-belief (Lekeas and Stamatopoulos 2014): a probability belief is an integer-to-probability vector function $f(s)$ (i.e., $f(s) \in R^{n-s}_+$, $\sum_{j=1}^{n-s} f_j(s) = 1$) for $s = 1, ..., n - 1$, defining the belief for all coalitions with $s$ members that outsiders are randomly partitioned into $j$-coalitions with a probability

\(^9\) One future research topic is to combine the efficient- and $j$-beliefs: replace $\Delta^*_j(s)$ in (15) with $\Delta^e_{j(S)}$, or the outsiders’ efficient partition among all $\Delta_{j(S)}$.\(^9\)
\( f_j(s), j = 1, ..., n-s, \) so the insiders’ payoff and the \( f \)-coalitional game are

\[
v_f(S) = \sum_{j=1}^{n-s} f_j(s) \phi_S(\Delta_j), \quad \text{all } S; \text{ and } \Gamma_f = \{N, v_f(\cdot)\},
\]

where \( \Delta_j \) is any \( \Delta = \{S, T_1, ..., T_j\} \in \Pi(S), \) all of which yield the same \( \phi_S \) for \( S \).

**Definition 3** The \( \gamma \)-, \( \delta \)-, \( \alpha^* \)-, \( e \)-, \( j \)- and \( f \)-cores of the game (3) are, respectively, the core of the above coalitional games \( \Gamma_\gamma, \Gamma_\delta, \Gamma_{\alpha^*}, \Gamma_e, \Gamma_j \) and \( \Gamma_f \) in (11-16), which are precisely defined by \( C_\gamma = \text{Core}(\Gamma_\gamma), \ C_\delta = \text{Core}(\Gamma_\delta), \ C_{\alpha^*} = \text{Core}(\Gamma_{\alpha^*}), \ C_e = \text{Core}(\Gamma_e), \ C_j = \text{Core}(\Gamma_j), \text{ and } C_f = \text{Core}(\Gamma_f) \).

Note that for the constant \( j \)-belief such that \( j(s) \equiv 1 \), all \( s \), \( j \)-belief is the same as the \( \delta \)-belief, so \( v_j(S) = v_\delta(S) \) and \( C_j = C_\delta \). For the special \( j \)-belief such that \( j(s) = n-s \), all \( s \), \( j \)-belief is the same as \( \gamma \)-belief, so \( v_j(S) = v_\gamma(S) \) and \( C_j = C_\gamma \).

Subsection 3.3 reviews the existence results for the above cores in a set of games (3) that are derived from oligopoly models. However, no similar results are known in a general normal form game (2). The relationships among the refinements (such as which is the strongest) are also unknown, with the only exception of the obvious relation that the \( \alpha^* \)-core is the largest (i.e., \( C_k \subseteq C_{\alpha^*} \) holds for \( k = \gamma, \delta, e, j \) and \( f \)).

3 **Core and its refinements as candidates of monopoly solutions**

This section first reviews ten oligopoly models and the equilibrium expressions, with the purpose of aiding readers to extend the known results in standard Cournot models to the other nine models in future research. It then reviews the known core results, including its existence as a precondition for the involved horizontal merger, its convexity and its empirical studies. It next reviews the results on core refinements, and
at the end it lists seven large areas of future research in the core and its refinements in more advanced oligopoly models.

### 3.1 The equilibrium expressions in ten oligopoly models

This subsection reviews ten oligopoly models and the involved equilibria, which can be obtained using the inverse matrix $A^{-1}$ in equation (16) in Zhao (2012). Because most previous studies have focused on a symmetric linear Cournot oligopoly (Cournot 1838), which is a special case of model 9 in (26), there is a long way to go in extending the known results to the more general cases of model 9 and then to the other nine even more general models.

A linear multi-product oligopoly with $n$ differentiated goods is defined by three parts: 1) $n$ cost functions $C_k(q_k) = c_k q_k$, $k \in N = \{1, \ldots, n\}$; 2) a set of multi-product firms $H = \{1, \ldots, h\}$ or a partition $\Delta = \{S_1, S_2, \ldots, S_h\}$ of $N$, with each firm $i \in H$ producing $n_i = |S_i|$ products in $S_i \in \Delta$ ($1 \leq n_i \leq n$, $\sum_{j=1}^{h} n_j = n$); and 3) a demand (in price-setting) or inverse demand (in quantity-setting) function for each of the $n$ products, whose definitions are given below.

Let $p = (p_1, \ldots, p_n)\top = (p_S, p_{-S}) = \{p_S| S \in \Delta\} = \{p_S| j \in H\}$ be the vector of prices, with $p_k$ as the price of each good $k \in N$, $p_S = \{p_k| k \in S\}$ as the price vector of each firm $S \in \Delta$, and $p_{-S} = \{p_k| k \in N \setminus S\}$ as the price vector of all other firms. Similarly, $q = (q_1, \ldots, q_n)\top = (q_S, q_{-S}) = \{q_S| S \in \Delta\}$ denotes the vector of products. In a price-setting oligopoly, or simply Bertrand oligopoly, or more accurately Edgeworth-Bertrand oligopoly, the demand for each good $k \in S_i$

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10 See Amir (2017) for survey on related oligopoly equilibria.

11 This title is suggested in Shubik (1980), because it was Edgeworth (1881) who originated the price-setting idea in Bertrand (1883).
produced by each firm $S_i \in \Delta$ are

$$q_k(p) = q_k(p_{S_i}, p_{-S_i}) = V - \gamma_{kk}p_k + \gamma_i \sum_{m \in S_i \setminus \{k\}} p_m + \sum_{j \in H \setminus \{i\}} \gamma_{ij} \sum_{m \in S_j} p_m, \quad (17)$$

where $V > 0$ is demand size; $\gamma_{kk} > 0$, $\gamma_i, \gamma_{ij} = \gamma_{ji} \in (0, 1]$, $k \in N$ and $i \neq j \in H$ are the substitution parameters.\(^\text{12}\) Now, the profit for each firm $S \in \Delta$ is $\pi_S(p) = \pi_S(p_S, p_{-S}) = \sum_{k \in S} (p_k - c_k)q_k(p)$.

In a quantity-setting or Cournot oligopoly, the inverse demands for the products of each firm $S_i \in \Delta$ are

$$p_k(q) = \hat{V} - \hat{\gamma}_{kk}q_k - \hat{\gamma}_i \sum_{m \in S_i \setminus \{k\}} q_m - \sum_{j \in H \setminus \{i\}} \hat{\gamma}_{ij} \sum_{m \in S_j} q_m, \text{ all } k \in S_i, \quad (18)$$

where $\hat{\gamma}_{kk}, \hat{\gamma}_i, \hat{\gamma}_{ij} > 0$, $k \in N$ and $i \neq j \in H$ are the parameters, and $\pi_S(q) = \pi_S(q_S, q_{-S}) = \sum_{k \in S} (p_k(q) - c_k)q_k$ is the profit of each $S \in \Delta$.

 Strategic behavior assumes that each firm chooses a best response, or that it takes other firms’ choices as given and chooses its choices to maximize its profit. In price-setting, a strategic equilibrium or noncooperative solution or Bertrand-Nash equilibrium (Bertrand 1883, Nash 1950) is a price vector $p^* = \{p^*_S| S \in \Delta\}$ such that each $p^*_S$ solves $Max \{\pi_S(p_S, p^*_{-S})| p_S \geq 0\}$, which is (under usual conditions) the solution of the following $h$ sets of first-order conditions:

$$\frac{\partial \pi_{S_i}(p)}{\partial p_k} = 0, \text{ all } k \in S_i \text{ and for each } S_i \in \Delta, \text{ or } Bp = d, \quad (19)$$

where $B = B_{n \times n}$ is partitioned into $h^2$ submatrices.\(^\text{13}\)

\(^{12}\) Note that internal substitution within a firm $i$ has identical rate $\gamma_i$ (i.e., between any $m$ and $t \in S_i$), and external substitution between two firms $i \neq j$ has identical rate $\gamma_{ij}$ among all of their products (i.e., between any $m \in S_i$ and $t \in S_j$). Even with such simplifications, the model is already complicated enough such that it is insolvable or precisely that the inverse of the matrix $B$ in (19) is unknown and remains as an open problem.

\(^{13}\) The block structure of $B$ in (19) follows by rearranging the first order conditions as

$$2\gamma_{kk}p_k - 2\gamma_i \sum_{m \in S_i \setminus \{k\}} p_m - \sum_{j \in H \setminus \{i\}} \gamma_{ij} \sum_{m \in S_j} p_m = V + \gamma_{kk}c_k - \gamma_i \sum_{m \in S_i \setminus \{k\}} c_m$$

for all $k \in S_i$ and each $S_i \in \Delta$. $B$ contains $[n + h(h + 1)/2 - h_1]$ constants, where $h_1$ is the number of singleton coalitions (i.e., single-product firms) in $\Delta$. See equation (1) in Zhao (2012) for details.
In quantity setting, a strategic equilibrium or Cournot-Nash equilibrium (Cournot 1838, Nash 1950) is an output vector \( q^* = \{ q^*_S | S \in \Delta \} \) such that each \( q^*_S \) solves \( Max\{\pi_S(q_S, q^*_{-S}) | q_S \geq 0 \} \), or the solution of these first-order conditions:

\[
\frac{\partial \pi_S(q_S, q_{-S})}{\partial q_k} = 0, \text{ all } k \in S \text{ and each } S \in \Delta, \text{ or } Bq = \bar{d},
\]

(20)

where \( B \) has the same structure of \( B \) in (19).

Keep in mind that \( A_2 \) (i.e., players are unable to take any form of coordinated or collective actions) is implicitly assumed behind the multi-product equilibria in (19-20) or the single-product equilibria in (21-22). If (19-20) are treated as the postmerger equilibria discussed below, then \( A_4 \) (i.e., firms within each merger or players in each coalition \( S \in \Delta \) are able to costlessly negotiate and enforce a joint action but players in all other coalitions \( T \notin \Delta \) are unable to take any form of coordinated or collective actions) or its variations are implicitly assumed.

It is convenient to call the Bertrand or Cournot equilibrium in single-product oligopolies (i.e., \( h = n \) in (17) and (18)) a premerger equilibrium. Precisely, a premerger Bertrand equilibrium \( p^0 = \{ p^0_i | i \in N \} \) satisfies

\[
p^0_i \in \text{ArgMax}\{\pi_i(p_i, p^0_{-i}) | p_i \geq 0 \}, \text{ all } i,
\]

(21)

and a premerger Cournot equilibrium \( q^0 = \{ q^0_i | i \in N \} \) satisfies

\[
q^0_i \in \text{ArgMax}\{\pi_i(q_i, q^0_{-i}) | q_i \geq 0 \}, \text{ all } i.
\]

(22)

In this regard, the general Bertrand equilibrium or the solution \( p = B^{-1}d \) of (19) (assuming the inverse exists) can be called the postmerger equilibrium for \( \Delta \). This
leads to three kinds of mergers given by the following models 1–3, respectively:\(^{14}\)

1) \[ \pi_S(p) = \sum_{k \in S} (p_k - c_k) q_k(p), \text{ all } S \in \Delta; \]
2) \[ \pi_S(p) = \sum_{k \in S} (p_k - c_S) q_k(p), \text{ all } S \in \Delta; \]
3) \[ \gamma_i = \text{Max}\{\gamma_{km} | k \neq m \in S_i\}, \gamma_{ij} = \text{Max}\{\gamma_{km} | k \in S_i, m \in S_j\}; \quad (23) \]
4) \[ q_k(p) = V - p_k + \gamma \sum_{m \neq k} p_m, \text{ all } k \in N; \]
5) \[ q_k(p) = V - p_k - \gamma (p_k - \bar{p}), \text{ all } k \in N, \]

where \( c_S = \text{Min}\{c_k | k \in S\} \) in model 2 is the smallest marginal cost of each merger or each multi-product firm \( S \in \Delta \), and \( \bar{p} = (\Sigma p_k)/n \) in model 5 is the average price.

In model 1, the profits of each merger \( S \in \Delta \) are simply the sum of its members’ profit (i.e., \( \pi_S(p) = \sum_{k \in S} \pi_k(p) \)), so it represents a set of \textit{simultaneous mergers without synergy}, from the premerger equilibrium \( p^0 \) in (21). On the other hand, model 2 represents \textit{simultaneous mergers with weak cost-synergy}, because each merger \( S \in \Delta \) can use its smallest marginal cost \( c_S = \text{Min}\{c_k | k \in S\} \) in producing all its outputs, or precisely \( \pi_S(p) = \sum_{k \in S} (p_k - c_S) q_k(p) \). Finally, model 3 represents \textit{simultaneous mergers with marketing-synergy}, because marketing outcomes such as

\[ \gamma_i = \text{Max}\{\gamma_{km} | k \neq m \in S_i\} \]

increases the demands for internal products, and

\[ \gamma_{ij} = \text{Max}\{\gamma_{km} | k \in S_i, m \in S_j\} \]

reduces the demand for competitors’ products.

Model 4 in (23) is called Dixit demand (1979) and is a special case of (17) when \( h = n \) (or there is no \( \gamma_i \)), \( \gamma_{kk} = 1 \) and \( \gamma_{ij} = \gamma \) (i.e., \( i \neq j, k \in N \)); model 5 is called Shubik demand (1980) and is another special case of (17), when \( h = n, \gamma_{kk} = [n+(n-1)]/n, \)

\(^{14}\) To facilitate future studies, the models are arranged in the same order of Zhao (2012), which can be expanded to include strategic complements surveyed in Vives (2017).
\( \gamma_{ij} \equiv \gamma/n, \ i \neq j, k \in N. \) Thus, the Dixit and Shubik demands are two models of premerger Bertrand equilibria.

The quantity-setting or Cournot equivalents of models 1 – 5 in (23) are:

6) \( \pi_{S}(q) = \sum_{k \in S} (p_{k}(q) - c_{k})q_{k}, \ \text{all} \ S \in \Delta; \)

7) \( \pi_{S}(q) = \sum_{k \in S} (p_{k}(q) - c_{S})q_{k}, \ \text{all} \ S \in \Delta; \)

8) \( \tilde{\gamma}_{i} = \min \{ \tilde{\gamma}_{km} | k \neq m \in S_{i} \}, \ \tilde{\gamma}_{ij} = \tilde{\gamma}_{ji} \equiv \max \{ \tilde{\gamma}_{km} | k \in S_{i}, m \in S_{j} \}; \)

9) \( p_{k}(q) = \hat{V} - q_{k} - \tilde{\gamma}_{m \neq k}q_{m}, \ \text{all} \ k \in N; \)

10) \( p_{k}(q) = \hat{V} - q_{k} + \tilde{\gamma}(q_{k} - \bar{q}), \ \text{all} \ k \in N, \)

where \( \bar{q} = \sum q_{j}/n \) in model 10 is the average output, and the synergy in model 8 reduces (increases) the negative effects of an output increase on own (rivals’) profits.

The details of (26) are similar to those of (23) and are thus skipped.

It is useful to note the following three remarks. First, most Cournot models in (26) and Bertrand models in (23) can be inverted from each other. For example, inverting model 5 in (23) yields the inverse demand in model 10 in (26), with \( \hat{V} = V \) and \( \tilde{\gamma} = \gamma/(1 + \gamma). \) Second, the Shubik demand or model 5 has an intuitive interpretation: a firm \( k \) that charges more (less) than the average price will be penalized (rewarded) by an amount equal to \( \gamma |p_{k} - \bar{p}|. \) This intuition is the reason why the Shubik demands or models 5 and 10 are used in the three mergers in (23) and (26) (and in Lemma 3). If one uses the Dixit demands or model 4 and model 9, one will get six additional merger models. Thus (23) and (26) actually provide a total of 16 oligopoly models (eight models each in both Cournot and Bertrand competitions).\(^{15}\)

Third, there are two main reasons why the supermajority of previous studies\(^{15}\) Dixit demand has the advantage of being the solution to a simple utility maximization problem. Let \( I_{n \times n} \) be the identity matrix, \( E_{n \times n} \) the matrix of ones, \( G = (1 - \gamma)I_{n \times n} + \gamma E_{n \times n}, \) and \( U(q, y) = y + V \sum q_{m} - q^{\top}Gq/2 \) the utility, where \( y \) is a composite measure of all other consumptions. Then, \( \max \{ U(q, y) | p^{\top}q + y \leq Y \} \) yields the inverse version of model 4 in (23), or model 9 in (26), where \( Y \) is fixed income. In addition, the consumer surplus is equal to \( CS = q^{\top}Gq/2. \)
have focused on quantity competition or Cournot oligopoly: 1) The expressions of known Bertrand equilibria are in general more involved and less tractable than the Cournot equilibria, and 2) the expressions of many Bertrand equilibria are unknown. Although the inverse $B^{-1}$ in (19) is unknown in the general cases and remains as an open mathematical problem, the partial solution or the inverse $A^{-1}$ in equation (16) of Zhao (2012) is sufficient to yield the equilibrium $p = A^{-1}d$ in most linear oligopolies that are relevant for empirical or theoretical studies. This matrix $A =$

\[
A_{n \times n} = \begin{pmatrix}
A_{11} & \cdots & A_{1h} \\
\vdots & \ddots & \vdots \\
A_{h1} & \cdots & A_{hh}
\end{pmatrix}
\]  

(27)

has the same structure of $B$ in (19). It is a very small class of $B$ in that it reduces $n(n-1)/2$ constants in $B$ in (19) to only three constants in $A$ in (27): $A_{ii}, i = 1, ..., h,$ is an $n_i \times n_i$ square matrix whose diagonal entries are a constant $a$ and other entries a constant $-b$, and $A_{ij}$, all $i \neq j$, is an $n_i \times n_j$ matrices of a constant $-c$, or precisely $\gamma_{kk} \equiv a/2$, $\gamma_{i} \equiv b/2$ and $\gamma_{ij} \equiv c$ for all $k, i$ and $j$ in (17).

The next lemma provides, as an example of $p = A^{-1}d$, the postmerger Bertrand equilibrium without synergy for a single merger $S = \{1, ..., t\}$, or precisely the equilibrium for $\Delta = \{S, t+1, ..., n\}$ in model 1 in (23) with Shubik demand or model 5. Without loss of generality, assume 1) $c_1 \leq c_2 \leq ... \leq c_t$, so $c_S = c_1$; and 2) the following assumption (A0) holds. A0 guarantees a positive output for all firms at both premerger and postmerger equilibria.\[16\]

**A0 (Assumption 0):** For each $S = \{1, ..., t\}$,

\[
\frac{nV + (n + (n-t)\gamma)\bar{c}_S + \gamma(n-t)\bar{c}_{-S}}{(2n + (2n-2t)\gamma)} > \bar{c}_S
\]

(28)

holds, where $\bar{c}_S = \Sigma_{i \in S}c_i/t$ and $\bar{c}_{-S} = \Sigma_{j \notin S}c_j/(n-t)$.

\[16\] This is an extension of the conditions in a single-product Cournot oligopoly in Zhao (2001a). See Pham Do and Folmer (2003) for discussion and Zhao (2009a, 377) for an application.
Lemma 3 Let \( p^* \) be the postmerger equilibrium for \( S = \{1, \ldots, t\} \) in model 1 with Shubik demand, \( c_S \) and \( c_{-S} \) be given in (28). Then, for each \( k \in S \), \( j \in N \setminus S \),

\[
\begin{align*}
p_k^* &= \frac{n(2n(1+\gamma) - \gamma)V}{\omega_0} + \frac{\gamma^2 t (n-t) c_S}{2\omega_0} + \frac{(n-t)\gamma(n(1+\gamma) - \gamma)c_{-S}}{\omega_0} + \frac{c_k}{2}, \\
p_j^* &= \frac{n(2n(1+\gamma) - t\gamma)V}{\omega_0} + \frac{\gamma^2 t (n(1+\gamma) - t\gamma)c_S}{2\omega_0} \\
&\quad + \frac{\gamma(n-t)(n(1+\gamma) - \gamma)(2n(1+\gamma) - t\gamma)c_{-S}}{(2n(1+\gamma) - \gamma)\omega_0} + \frac{(n(1+\gamma) - \gamma)c_j}{2n(1+\gamma) - \gamma},
\end{align*}
\]

where \( \omega_0 = \gamma^2 (n-t) (t + 2n - 2) + 2n\gamma (3n - t - 1) + 4n^2 \).

The above expressions become the premerger equilibrium when \( t = 1 \) in (29); the postmerger equilibrium with weak synergy in model 2 when \( c_k = c_1 = c_s \), all \( k \in S \); and the postmerger equilibrium with Dixit demand when \( V \) in (29) is replaced by \( V/(1 + \gamma - n\gamma) \) and \( \gamma \) by \( n\gamma/(1 + \gamma - n\gamma) \).\(^{17}\)

### 3.2 The core in oligopoly TU games

A homogeneous Cournot oligopoly is given by an inverse demand \( p(\Sigma q_i) \) and \( n \) cost functions \( C_i(q_i) \), \( 0 \leq q_i \leq z_i \), with \( z_i > 0 \) as firm \( i \)'s capacity, or by a normal form game \( \Gamma = \{N, X_i, u_i\} \) in (2) in which

\[
u_i = \pi_i(q) = p(\Sigma q_j)q_i - C_i(q_i), \quad X_i = [0, z_i], \quad \text{all } i.
\]

Under the usual conditions of a Cournot oligopoly such as decreasing demand and continuity, both weak and strong separability in Lemma 2 hold because the outsiders’ choices in both \( v_{\alpha}(S) \) in (6) and \( v_{\beta}(S) \) in (7) are equal to their full capacity at \( x_{-S} = q_{-S} = z_{-S} = \{z_j\mid j \notin S\} \). This implies \( v_{\alpha}(S) = v_{\beta}(S) \) and \( \text{Core}(\Gamma_\alpha) \)

\(^{17}\) This follows from \( q_k = V - p_k + \gamma \Sigma_{m \neq k} p_m = (1 + \gamma - n\gamma) \left[ \frac{V}{1 + \gamma - n\gamma} - p_k - \frac{n\gamma(p_k - p)}{(1 + \gamma - n\gamma)} \right] \), and the observation that the first term \((1 + \gamma - n\gamma)\) does not enter the first-order conditions. See footnote 9 in Zhao (2012) for details.
Core = \text{Core}(\Gamma_\beta). Therefore, there is no need to make the \(\alpha\)- and \(\beta\)-distinction in oligopoly models, and one can simply use the term core, which will be non-empty under the additional assumption that each \(\pi_i(q)\) is concave in \(q = (q_1, \ldots, q_n)\). Such results are first obtained by the author and are given here.

**Proposition 1** (Zhao 1999a, 160) Let \(C_\alpha\) and \(C_\beta\) be the \(\alpha\)- and \(\beta\)-cores of an oligopoly \(\Gamma\) in (30). Assume \(p(\sum q_j)\) is decreasing and each \(\pi_i(q)\) is continuous. Then, i) \(C_\alpha = C_\beta = C(\Gamma)\); and ii) \(C(\Gamma) \neq \emptyset\) if each \(\pi_i(q)\) is concave.

Although concavity in part (ii) is a strong condition,\(^{18}\) it can be weakened in large classes of oligopolies such as the linear version of (30) in (34) in subsection 3.2.2.

### 3.2.1 Non-empty core as a precondition for horizontal mergers

A fundamental role of the core theory in industrial organization is that non-empty core is a precondition for horizontal mergers. This argument is summarized in the next proposition. For simplicity, define a monopoly merger contract in an oligopoly (30) as a triplet \((N, \bar{q}, \theta)\) of the set of firms \(N\), monopoly supply \(\bar{q}\) and a split of monopoly profits \(\theta\) (i.e., \(\bar{q} \in \text{ArgMax}\{\sum_{j=1}^{n} \pi_j(q)\mid q_j \in [0, z_j], \text{all } j\}, \theta \geq 0\) and \(\Sigma\theta_j = \Sigma\pi_j(\bar{q}) = \pi^m = v(N)\)).

**Proposition 2** (Two preconditions for the monopoly merger, Zhao 2009a, 378) Let \(q^0\) be the premerger equilibrium in (30), \(C(\Gamma)\) its core, and \((N, \bar{q}, \theta)\) an observed monopoly merger. Then, i) \(\theta_j \geq \pi_j(q^0)\), all \(j\); and ii) \(\theta \in C(\Gamma)\).

Part (i) is the well-known profitability precondition (or incentive to merge), and part (ii) is the non-empty core precondition. The merger would have made at least

\(^{18}\) Continuity can also be weakened, in the same manner of Uyanik (2015) on the TU \(\alpha\)-core in a normal form game (2).
one firm worse off (i.e., a firm $j$ gets less than its premerger profits $\pi_j^0$) if part (i) fails, and at least one coalition worse off (i.e., a coalition $S$ gets less than its guaranteed or worst profits $v(S)$) if part (ii) fails. Therefore, the failure of either precondition will violate a firm’s or a coalition’s rationality, so both must hold in successful mergers.

Keep in mind that these are necessary, rather than sufficient, conditions for a monopoly merger. Failing either or both will result in a merger failure, and meeting both will not guarantee a merger success. In addition, they are independent of each other: Example 1 in the next subsection reports a profitable monopoly with an empty core, and Example 2 an unprofitable monopoly with a non-empty core.

These preconditions make it possible to study how merging costs or the transaction costs of a merger affect merger formation and how to empirically estimate the sizes of such merging costs. Let $mc(S) \geq 0$ denote the merging cost of each merger $S \subseteq N$. For simplicity, assume $mc(S) = 0$ for all non-monopoly merger $S \neq N$, to focus on the monopoly merging cost $mmc = mc(N) \geq 0$. The next proposition shows that

$$mmc^* = \pi^m - Max\{\Sigma \pi_j^0, mnbp\}$$

and

$$mmc^0 = \pi^m - Min\{\Sigma \pi_j^0, mnbp\}$$

are, respectively, an upper bound of the merging cost for a successful monopoly merger and a lower bound for a failed or unobserved monopoly merger, where $\pi^m$ and $\pi_j^0$ are the same monopoly and premerger profits as in Proposition 2, and $mnbp$ is given in (5) for the oligopoly game (30).

**Proposition 3** (Zhao 2009a, 2009b) *Given a monopoly merger in (30), let $mmc^*$ and $mmc^0$ be given in (31-32). Then, i) $mmc \leq mmc^*$ if the merger is successful; and ii) $mmc > mmc^0$ if the merger is prevented by failed preconditions.*
3.2.2 The core with weak synergy

The following assumption (A0.1) modifies the concept of weak synergy in the oligopoly (30). Though the synergy such as $c_S$ in (23) and (26) might be quite large in reality, it is called *weak synergy* in Farrell and Shapiro (1990) for comparison with strong synergies involving economies of scale.

**A0.1**: (i) Each $\pi_i(q)$ is continuous in $q$ and quasi-concave in $q_i$, and $p(\Sigma q_j)$ is decreasing; (ii) all equilibria are positive and interior solutions; and (iii) the capacity and cost function for each merger $S$ are

$$z_S = \Sigma_{j \in S} z_j, \quad C_S(y) = \min \{ \Sigma_{j \in S} C_j(q_j) | y = \Sigma_{j \in S} q_j \leq z_S, q_S \geq 0 \}. \quad (33)$$

Most results reviewed in this chapter deal with a linear (30) or a subset of model 9 with $\gamma = 1$ and capacities in (26): $p(\Sigma q_j) = a - \Sigma q_j, \quad C_i(q_i) = c_i q_i, \quad q_i \in [0, z_i]$, which can be given by a $(2n+1)$-vector

$$(a, c, z) \in R^{2n+1}_+, \quad \text{where } c = (c_1, ..., c_n) \text{ and } z = (z_1, ..., z_n) \quad (34)$$

are the vectors of marginal costs and capacities, and $a > 0$ is the intercept of the demand. Without loss of generality, assume $c_1 \leq ... \leq c_n < a$. Then, above A0.1 becomes for all $S$, $C_S(y) = c_S y$, $y \leq z_S = \Sigma_{j \in S} z_j$,

$$0 < (a - c_S - z_S)/2 \leq z_S \quad \text{and} \quad (a - c_1)/2 \leq z_N = \bar{z} = \Sigma_{j \in S} z_j, \quad (35)$$

where $z_S = \Sigma_{j \notin S} z_j$, and $c_S = \min \{ c_k | k \in S \}$ is the same as in (23) and (26).

A symmetric linear Cournot oligopoly is the case when $c_i = c$ and $z_i = z$, all $i$, or $(a, c, z) \in R^3_+$.\footnote{The same letter $c$ is used here as a scalar and in (34) as a vector. This should cause no confusion because the meaning will be clear in the contexts. Similar simplification holds for letter $z$.} In such symmetric cases, the conditions in (35) become

$$0 < (a - c)/(n + 1) \leq z \leq (a - c)/(n - 1), \quad (36)$$
which leads to, as shown in Zhao (2009a), \( mnbp = n(a-c-z)^2/[4(n-1)] < v(N) = (a-c)^2/4 \), so the core is not only non-empty but also has a non-empty (relative) interior. Such core results are summarized in the next proposition.

**Proposition 4** (Zhao 2009a, 381) Let \( mmc \geq 0 \) be the monopoly merging cost in (34), \( C(\Gamma) \) its core, and assume parts (ii-iii) of A0.1 or (35). Then, i) the core has a non-empty (relative) interior if \( mmc = 0 \); ii) \( C(\Gamma) \neq \emptyset \) if

\[
mmc \leq (a-c_1)^2/4 - \{n(a-c_1-z_{\text{min}})^2/[4(n-1)]\}; \quad \text{and} \tag{37}
\]

and

\[
iii) \text{ in symmetric case with } c_i = c \text{ and } z_i = z, \text{ all } i, C(\Gamma) \neq \emptyset \text{ if and only if}
mmc \leq (a-c)^2/4 - \{n(a-c-z)^2/[4(n-1)]\}, \tag{38}
\]

where \( c_1 = \text{Min}\{c_k \mid k \in N\} \) and \( z_{\text{min}} = \text{Min}\{z_k \mid k \in N\} \) are the minimal marginal cost and minimal capacity.

The core’s interior has important implications in empirical studies. In the event of small shocks to the market, a core with a non-empty interior remains non-empty, but a non-empty core with an empty interior could become empty. Thus, a long-lived merger or trust suggests that the core has a non-empty interior, and short-lived ones suggest either an empty-core or a non-empty core with an empty interior, which are the causes for both merger failure and the breakup of completed mergers such as the breakup of AOL-Time Warner.

The following two examples (Zhao 2009a) show the independence of the two merger preconditions; they also illustrate part (iii) in Proposition 4.

**Example 1** \( n = 3, (a,c,z) = (6,0.8,1.5) \); or \( p = 6 - \sum x_j, C_i(x_i) = 0.8x_i, 0 \leq x_i \leq 1.5, i = 1, 2, 3 \); and \( mmc = 1.65 \). The premerger and monopoly profits are \( \pi_i^0 = 1.69, \pi^m = 6.76, \) and \( mnbp = 5.13 \). By \( v(123) = \pi^m - mmc = 5.11 \)
\[ \sum \pi_i^0 = 5.07, \text{ the merger is profitable.}^{20} \text{ By (38) and by } \text{mmc} = 1.65 > (a - c)^2/4 - n(a - c - z)^2/[4(n - 1)] = \pi^m - \text{mnbp} = 1.63, \text{ the core is empty.} \]

**Example 2** \( n = 3, p = 6 - \Sigma x_i, C_i(x_i) = 0.5x_i, 0 \leq x_i \leq 2, \text{ all } i, \text{ and } \text{mmc} = 2. \) The premerger and monopoly profits are \( \pi_i^0 = 1.89, \pi^m = 7.56, \text{ and } \text{mnbp} = 4.59. \) The core is non-empty because \( \text{mmc} = 2 < \pi^m - \text{mnbp} = 2.97, \text{ and the merger is not profitable because } v(123) = 5.56 < \sum \pi_i^0 = 5.67. \)

The next proposition shows how excessive capacity affects the estimated bound of monopoly merging costs. Let \( \tau \geq 0 \) be the rate of excessive capacity as defined in

\[ z = (1 + \tau)(a - c)/(n + 1), \]  

so \( \tau = 0 \) means full capacity at premerger equilibrium: \( q_i^0 = (a-c)/(n+1) = z, \text{ all } i. \)

**Proposition 5** (Zhao 2009a, 383) Let \( \tau_1 = n-2\sqrt{n-1}, \text{ and } \text{mmc}^* \text{ and } \tau \text{ be given in (31) and (39). Then, } \text{mmc}^* = n(a - c)^2(n - 1)^2/[4n(n + 1)^2] \text{ if } \tau \geq \tau_1, \text{ and } \text{mmc}^* = \frac{n(a - c)^2(n - 1)(n + 1)^2 - n(n - \tau)^2}{4n(n - 1)} \text{ if } \tau < \tau_1. \]

The above results indicate that a larger capacity will strengthen the non-empty core precondition, so the monopoly merger is more likely to be formed. This is consistent with and thus provides a new explanation for the stylized fact that mergers are likely to occur in markets plagued by excess capacities.

### 3.2.3 The convexity in oligopoly games

Convex games or supermodular set functions are interesting in both economics and mathematics and have generated a large literature. Only the less technical results

\[ ^{20} \text{Coalition } \{1,2,3\} \text{ is simplified as 123. Similar simplifications hold in other places where no confusion arises.} \]
in oligopoly TU games are reviewed here. For non-technical readers, it is sufficient to know three conclusions in a convex oligopoly game: 1) a convex game exhibits increasing returns to scale in coalition size or the property that each i’s marginal contribution to a coalition increases as the coalition expands, so there is an incentive to get larger and eventually form the grand coalition; 2) the core is non-empty; and 3) both the nucleolus (Schmeidler 1969) and Shapley value (Shapley 1953) are perfect answers to the question of how to split the monopoly profits.\footnote{See Driessen and Meinhardt (2010), Meinhardt (2002, 2013), Vives (2017) and Zhao (1999b) for reviews. In such cases, the nucleolus coincides with both the pre-kernel and kernel and thus satisfies additional nice properties. See Meinhardt (2013, 32) for more discussion.}

**Definition 4** The game $\Gamma$ in (1) or (9) is convex if for any $S, T \subseteq N$,

$$v(S) + v(T) \leq v(S \cap T) + v(S \cup T).$$

Assume the conditions in (35) for a linear oligopoly (34), one has $v(S) + v(T) \leq v(S \cup T)$ for any $S \cap T = \emptyset$ (see Theorem 1 in Zhao 1999b), or that the oligopoly game $\Gamma$ for (34) is superadditive. Because convexity in (41) implies the preceding superadditivity, convex games are stronger than superadditive games.

The first main result in convex oligopoly games is a necessary and sufficient condition for an oligopoly (34) reported in Zhao (1999b),\footnote{The linear model in Zhao (1999b) contains fixed costs and is given by a $(3n + 1)$-vector $(a, c, d, z) \in R_{+}^{3n+1}$, or $p(\Sigma q_j) = a - \Sigma q_j, \ C_i(q_i) = d_i + c_i q_i, q_i \in [0, z_i]$, with $d = (d_1, ..., d_n)$ as the vector of fixed costs and $(a, c, z)$ the same as in (34). Because fixed costs have no effects on convexity, this review sticks with $(a, c, z)$ or assumes $d = 0$.} which has been extended along several directions in Norde, Pham and Tijs (2002), Driessen and Meinhardt (2005, 2010), Lardon (2010), and Hou, Driessen and Lardon (2011). Let

$$\Omega = \{(S, T, i)\mid S \subset T \subset N, i \in N \setminus T \text{ and } c_S - c_{S \cup i} > c_T - c_{T \cup i}\}$$

\footnote{A coalition’s marginal costs exhibit supermodularity if $c_{S \cup i} - c_S \leq c_{T \cup i} - c_T$ for $S \subset T$ and $i \notin T$. Thus, $(S, T, i) \in \Omega$ in (42) implies strict supermodularity or $c_{S \cup i} - c_S < c_{T \cup i} - c_T$.} denote the set of coalitions whose marginal costs exhibit strict supermodularity.\footnote{It is not difficult to show that the game is convex if $\Omega = \emptyset$ (see Theorem 2 in Zhao}
If $\Omega \neq \emptyset$, for each $(S, T, i) \in \Omega$, define
\[
f(S, T, i) = c_S^2 - c_{S\backslash i}^2 - (c_T^2 - c_{T\backslash i}^2) + 2z_i(c_S - c_T + \sum_{j \in T \setminus S} z_j) + 2[(c_S - c_{S\backslash i})\sum_{j \notin S, j \neq i} z_j - (c_T - c_{T\backslash i})\sum_{j \notin T, j \neq i} z_j],
\]

\[
F(S, T, i) = f(S, T, i)/[2(c_S - c_{S\backslash i} - (c_T - c_{T\backslash i}))],
\]
and
\[
\omega = \text{Min}\{F(S, T, i)|(S, T, i) \in \Omega\}. \quad (44)
\]

Under the conditions in (35), one has $\omega > 0$ (see Lemma 5 in Zhao 1999b). Although the economic meaning of $\omega$ is still not well understood, it nevertheless fully characterizes the convexity.

**Proposition 6** (Zhao 1999b, 195) Given $(a, c, z)$ in (34) and $\omega$ in (44), assume (35).

Then, $\Gamma$ in (9) is convex if and only if $a \leq \omega$.

**Example 3** (Zhao 1999b) $n = 3, (a, c, z) = (7, 4, 2.25, 2.25, 1.3, 1.3, 1.3)$. One has $\omega = 6.6907, v(1) = 0.04, v(2) = v(3) = 1.1556, v(12) = v(13) = v(23) = 2.9756, v(123) = 5.6406$. By Proposition 6 and $\omega < a = 7$, the game is not convex. Indeed, (41) fails for $S = 12, T = 13$: $v(S) + v(T) = 5.9512 > v(S \cup T) + v(S \cap T) = 5.6806$. Let $a$ be decreased to $a = 6.65$, all other parameters be unchanged, one has the same $\omega = 6.6907$ and new coalitional values: $v(1) = 0.0006, v(2) = v(3) = 0.81, v(12) = v(13) = v(23) = 2.4026$ and $v(123) = 4.84$. By $a = 6.65 < \omega$, the game is now convex. Indeed, one can verify that (41) holds for all $S$ and $T$.

Norde, Pham and Tijs (2002) extend the above results to mergers without weak synergy (or *without transferable technologies*). They show that such oligopoly games are always convex. Precisely, let $\pi_S(q_S, z_{-S}) = \sum_{k \in S}[p(\sum_{j \in S} q_j + \sum_{j \notin S} z_j) - c_k]q_k$ be
the same as in model 6 in (26), and a coalition’s payoff be given by

\[ v(S) = v_\alpha(S) = v_\beta(S) = \text{Max} \{\pi_S(q_S, z_{-S})|q_k \in [0, z_k], \text{ all } k \in S\}. \quad (45) \]

**Proposition 7** (Norde, Pham and Tijs 2002, 203) *Given \((a, c, z)\) in (34), let \(v(S)\) in \(\Gamma\) in (9) be given in (45). Then \(\Gamma\) is convex.*

Driessen and Meinhardt (2005) reestablish Proposition 7 using a new technique that has an economic interpretation. This effective technique allow them to obtain more general convex games with weak synergy without assuming interior solutions. This is summarized in the next proposition.

**Proposition 8** (Driessen and Meinhardt 2010, 330) *Given \((a, c, z)\) in (34), assume part (iii) of A0.1, and let \(\Omega\) and \(\omega\) be defined as the special case of \(\Omega\) and \(\omega\) in (42-44) when \(T = S \cup j\) for all \(j \notin T \cup i\). Then, \(\Gamma\) is convex if \(a \geq \omega\).*

Lardon (2010) extends Propositions 1 and 7 to Bertrand competition and establishes this result: In a symmetric linear Shubik model or in a symmetric model 5 in (23), the \(\alpha\)-core and \(\beta\)-core are identical and convex. Hou, Driessen and Lardon (2011) give further extensions that similar results hold in asymmetric Shubik model under reasonable assumptions.

### 3.3 Refinements of the core

Refining the core in an oligopoly provides deeper understandings about monopoly stability and helps search for the sufficient conditions of monopoly formation. Such refinements are achieved in two steps: 1) As Lekeas (2013, 2) puts it, "Convert the oligopoly (or normal form game) to a partition function game in Thrall and Lucas (1963) by computing the quasi-hybrid solution (each coalition chooses an efficient..."
solution, given others’ choices) for each partition in Zhao (1991); and 2) Study one of the core solutions defined in Definition 3 or other core solutions for the converted partition function game (3).\textsuperscript{24}

Conversions in the first step implicitly assume that the following simpler version of $A4$ holds for each partition:

**A4.0 (Assumption 4.0):** Given a partition $\Delta = \{S_1, S_2, ..., S_h\}$, $A1$ holds for each $S \in \Delta$, and $A2$ holds for each $T \notin \Delta$ such that there are $i \neq j$, $T \cap S_i \neq \emptyset$ and $T \cap S_j \neq \emptyset$.

Because each coalition $S \in \Delta$ in (19) or (20) chooses an efficient solution, the same as that under $A4.0$, the solutions in (19-20) have been called quasi-hybrid solutions (Zhao 1991), as compared with hybrid solutions\textsuperscript{25} under $A4$ in which each $S \in \Delta$ chooses a core solution. Such quasi-hybrid solutions are, as pointed out in Zhao (1991), the same as the noncooperative solution of Shapley (1956, 1959) for the following multiobjective game ($MOG$):

$$\Gamma_m = \{H, Y_j, v_j\},$$

where $H = \{1, ..., h\} =: \{S_1, S_2, ..., S_h\}$ is the set of new players (i.e., new names for coalitions in $\Delta = \{S_1, S_2, ..., S_h\}$), $Y_j = Y_{S_j} = \Pi_{i \in S_j} X_i$ (in (2) or (30)) is $j$’s choice set, and $v_j = v_{S_j} = \{u_i | i \in S_j\}$ in (2) (or $= \{\pi_i | i \in S_j\}$ in (30)) is $j$’s vector payoff function. Thus, the solutions in (19-20) should be cited either as a noncooperative solution of the $MOG$ (46) or a quasi-hybrid solution for (2) and (30).\textsuperscript{26}

\textsuperscript{24}As already discussed in subsection 2.3, an exception is the leader-follower core or $lf$-core (Currarini and Marini 2003), see Proposition 20 at end of this subsection for an $lf$–core result.

\textsuperscript{25}See Allen (2000, 147), Diamantoudia and Xue (2007, 108) and McCain (2008) for discussion about the significance and generality of hybrid solution.

\textsuperscript{26}Chander and Tulkens (1997) study a class of normal form games. They gave such quasi-hybrid equilibria a new name without citing Shapley (1956) or Zhao (1991). See Folmer and Mouche (1994) and Zhao (2016) for more discussions about the connection between $MOG$ and the solutions in (19-20).
Given a linear oligopoly in model 4 or 5 in (23) or model 9 or 10 in (26), the inverse $A^{-1}$ of $A$ in (27) readily yields eight classes of postmerger equilibria (i.e., using model 4 or 5 with model 1 or 2, and model 9 or 10 with model 5 or 6) for each partition $\Delta = \{S_1, S_2, ..., S_h\}$, and thus leads to eight partition function games, only one of which (i.e., symmetric linear Cournot with no synergy or models 6 and 9) is well studied. The following results are obvious and can be found in the discussions in Yong (2004) and Zhao (1996, or 2013).

**Lemma 4** Given $\Gamma_p$ in (3) for a symmetric $(a,c,z)$ in (34), assume $a > c$ and $z_i = z = \infty$. Let $C = C_\alpha = C_\beta$ be its core in (9), and $C_\gamma$, $C_\delta$, $C_{\alpha^*}$ and $C_e$ be its $\gamma$, $\delta$, $\alpha^*$- and $e$-cores in Definition 3. Then, $C_\delta \subseteq C_e \subseteq C_\gamma \subseteq C_{\alpha^*} \subseteq C$.

The first major refinement of the core in oligopoly games is Yong’s characterization of his $e$-core given below:

**Proposition 9** (Yong 2004, 10) Given $\Gamma_e$ in (14) for a symmetric $(a,c,z)$ in (34), assume $a - c > 0$ and $z_i = z = \infty$. Then, i) $\theta \in C_e = \text{Core}(\Gamma_e) \Leftrightarrow \sum \theta_j = v(N)$ and $\theta_j \geq v(j) \equiv v(1)$, all $j$; and ii) $C_e \neq \emptyset \Leftrightarrow n \leq 4$.

Thus, in a standard symmetric linear Cournot oligopoly, the $e$-core and imputation set are identical, monopoly can possibly be formed under the efficient-belief with four or fewer firms, and will not be formed under efficient-belief with five or more firms.

The next proposition is an extension to diseconomies of scale given by $C_k(q) \equiv C(q) = cq + dq^2$. By $MC/AC = (c + 2dq)/(c + dq) > 1$, average cost is increasing so there exists diseconomies of scale, and it is cheaper for a merger of $m$ firms to produce a smaller quantity $\sum_{k \in S} q_k/m$ at each of its $m$ plants than to produce the sum $\sum_{k \in S} q_k$ at one large plant (i.e., $C_S(\sum_{k \in S} q_k) = mC(\sum_{k \in S} q_k/m) < C(\sum_{k \in S} q_k)$).
**Proposition 10** (Yong 2004, 21-24) Given $p = a - \Sigma q_j$ and $C_k(q) = cq + dq^2$, all $k$; assume part iii) of A0.1 or $\pi_S(q) = p(\Sigma_{j \in S} q_j) - C_S(\Sigma_{k \in S} q_k)$, $S \in \Delta$ for all $\Delta$. Let $n^0 = [5 + 5d - 2d^2 + (1 + d)(9 + 2d)]/[2(1 + 2d)]$. Then, $C_e \neq \emptyset \iff n \leq n^0$.

Below is another interesting and non-trivial extension in Yong (2004), which studies a Bertrand-Shubik model with zero costs (i.e., model 5 in (23) with zero costs).

**Proposition 11** (Yong 2004, 14) Given $\pi_S(p) = \Sigma_{k \in S}p_k[V - p_k - \gamma(p_k - p)]$, $S \in \Delta$ for all $\Delta$, let $\gamma^0 = 2n[(n - 2)^2 - 3 + (n - 2)\sqrt{(n - 2)^2 + 3}]/[9(n - 1)]$. Then, $C_e \neq \emptyset \iff \gamma \geq \text{Max}\{0, \gamma^0\}$.

Yong (2004) shows that the conditions of Proposition 11 always hold if $n \leq 3$. Readers can find other extensions in Yong (2004) such as with capacity constraints. Proposition 11 appears to imply the $r$-core result in Huang and Sjostrom (2003):\textsuperscript{27}

**Corollary 1** (Huang and Sjostrom 2003, 208) In the same oligopoly of Proposition 11, $r$-core $\neq \emptyset \iff \gamma \geq \hat{\gamma}(n)$, where for $i = 1, \ldots, 7$, $\hat{\gamma}(i) = \text{Max}\{0, \gamma^0(i)\}$; $\hat{\gamma}(8) = 19, \hat{\gamma}(9) = 43.75$, and $\hat{\gamma}(n) = \infty$, all $n \geq 10$.

Lardon (2012) provides the second major contribution in refining the core.

**Proposition 12** (Lardon 2012, 403) Let $C_\gamma$ be the $\gamma$-core of (30). Assume $p(\Sigma q_j)$ is decreasing and each $\pi_i(q)$ is continuous and concave. Then, $C_\gamma \neq \emptyset$.

\textsuperscript{27} The concept of $r$-core in Huang and Sjostrom (2003) is confusing because it is not clear whether the cooperative or the non-cooperative approach to cooperation is used. The relation of $r$-core with other cores is unknown because none of the known $\alpha$- or $\beta$-core results are cited in the paper. Based on the facts that $\gamma^0(n)$ generates most of the values of $\hat{\gamma}(n)$, the author suspects that some values of $\hat{\gamma}(n)$ in Huang and Sjostrom (2003) (i.e., some $n \geq 8$) are incorrect, and conjectures that the $e$-core and $r$-core are identical. The author thanks Giorgos Stamatopoulos for bringing up this issue to his attention.
Thus, under the same conditions of Zhao (1999a) for a non-empty $\beta$-core in (30), $\gamma$-core is also non-empty. This implies Rajan’s $\gamma$-core result with 3 or 4 firms (1989):

**Corollary 2**  
(Rajan 1989, 871) Under the conditions of Proposition 12, $C_\gamma \neq \emptyset$ if $n = 3$ or 4.

The results in Proposition 12 have been extended to coalitional interval games in which the payoffs of each coalition are given by a closed interval (Lardon 2016).

Below is another interesting result in Lardon (2012).

**Proposition 13**  
(Lardon 2012, 406) Given $\Gamma_\gamma$ in (11) for (34) with symmetric $c \in R^n_+$ and asymmetric $z \in R^n_{++}$, assume part iii) of A0.1. Let $q^0$ be the premerger equilibrium, $\pi^m = v(N)$ the monopoly profit, and $\theta_j = v(N)q^0_j/\Sigma q^0_k$, all $j$. Then, $\theta \in C_\gamma = \text{Core}(\Gamma_\gamma)$.

Thus, in linear cases, proportional split of the monopoly profit by premerger market shares is in the $\gamma$-core. This implies a symmetric $\gamma$-core result in Currarini and Marini (2015):

**Corollary 3**  
(Currarini and Marini 2015, 11) In symmetric $(a,c,z)$ with A0.1, equal split of the monopoly profit is in the $\gamma$-core.

Currarini and Marini (2006) study a class of symmetric normal form games and provide two $\gamma$-core results with applications in oligopoly models. The next two propositions are the non-technical versions of their results. Readers are referred to their paper for the technical assumptions and details.

**Proposition 14**  
(Currarini and Marini 2006, Theorem 3.1, 119) Let $C_\gamma$ be the $\gamma$-core in a class of (2) in which $X_i \equiv X \subset R$, all $i \in N$. Then, $C_\gamma \neq \emptyset$ if 1)
the players are symmetric and $X$ is convex, and 2) all $u_i(x)$ exhibit monotone externalities and increasing differences.

This result seems to hold in classes of Bertrand oligopolies with strategic complementarity, but such claims need to be verified in future studies.

**Proposition 15** (Currarini and Marini 2006, Theorem 3.2, 122) Consider the same model of Proposition 14. Then, $C_\gamma \neq \emptyset$ if 1) the players are symmetric and $X$ is convex, and 2) all $u_i(x)$ are strictly quasi-concave, satisfy contraction property, and exhibit monotone externalities.

This proposition holds in standard linear Cournot models and thus implies the $\gamma$-core result in Corollary 3.

Watanabe and Matsubayashi (2013) study a differentiated linear Cournot model with three or four firms and give a positive $\gamma$-core result. Chander (2014) studies the noncooperative foundation of $\gamma$-core in (2) and provides a $\gamma$-core result in oligopoly (30). His main result, Theorem 7 in Chander (2014), appears to be identical to Proposition 12. It is not clear at the present whether concavity in profit functions can be removed while still maintaining a non-empty $\gamma$-core in the oligopoly (30).

Zhao (2013) gives several $\delta$-core results in a three-firm asymmetric linear Cournot oligopoly, one of which is given in the next proposition.\(^\text{28}\) Note that complication arises once asymmetry is allowed and many of the intuitions in symmetric models no longer hold. Given $(a, c, z) = (a, c_1, c_2, c_3, z_1, z_2, z_3) \in R^7_+$, with $c_1 \leq c_2 \leq c_3$ (so firm 1 is the most efficient, and 3 the least efficient), define

\[
\varepsilon_2 = (c_2 - c_1)/(a - c_1), \quad \varepsilon_3 = (c_3 - c_1)/(a - c_1),
\]

\[
\omega_1(\varepsilon_2) = [2 - \sqrt{1 + 8\varepsilon_2^2 - 20\varepsilon_2^3}]/4,
\]

\(^{28}\)Gabszewicz, Marini and Tarola (2016) give a $\delta$-core result in vertically differentiated markets with $n$ firms.
where $\varepsilon_2$ and $\varepsilon_3$ represent the (relative) cost advantages of firm 1 over 2 and 3. The larger the value of $\varepsilon_i$, the less efficient (or smaller) the firm $i$. Obviously, $\varepsilon_2 = \varepsilon_3 = 0$ is the symmetric case, and it is easy to check that $0 \leq \varepsilon_2 \leq \varepsilon_3 \leq 0.5$ holds. These two intermediate parameters simplify the original and impossible task of characterizing the $\delta$-core in seven dimensions to a manageable though still difficult task of characterizing the $\delta$-core in only two dimensions.

**Proposition 16** (Zhao 2013, 12) Given $(a, c, z) \in R^3_+$, assume parts ii-iii) of A0.1, and let $\varepsilon_2, \varepsilon_3$ and $\omega_1(\varepsilon_2)$ be given in (47-48). Then, i) $C_{\alpha^*} \neq \emptyset$, and ii) $C_\delta \neq \emptyset \iff \varepsilon_3 \geq \omega_1(\varepsilon_2)$, and $\varepsilon_3 \geq \omega_1(\varepsilon_2)$ holds if $\varepsilon_2 \in [1/6, 1/2]$.

Thus, monopoly is always $\alpha^*$-stable or can possibly be formed under the $\alpha^*$-belief; it can possibly (will not) be formed under the $\delta$-belief if firms 2 and 3 are sufficiently small, e.g., $\varepsilon_3 \geq \varepsilon_2 \geq 1/6$ (sufficiently large, e.g., $\varepsilon_3 < \omega_1(\varepsilon_2)$). In particular, it will not be formed under the $\delta$-belief in symmetric case (i.e., $\varepsilon_3 = \varepsilon_2 = 0$). The next corollary shows that monopoly is both $\delta$-stable and socially optimal if there are large cost-savings. Here, optimality is in the sense of second best, which has the maximal welfare (= total profits + consumer surplus) among the five partitions.

**Corollary 4** (Zhao 2013, 12) If $\varepsilon_2 \geq 5/22$, monopoly is both $\delta$-stable and optimal.

Currarini and Marini (2015) give a negative $\delta$-core result in symmetric linear Cournot oligopolies.

**Proposition 17** (Currarini and Marini 2015, 12) In symmetric $(a, c, z)$ in (34) with $z = \infty$, $C_\delta = \emptyset$.

This negative result is consistent with the symmetric case of Proposition 16.

Lekeas (2013) studies a differentiated symmetric linear Cournot oligopoly or a general model 9 in (26) and provides existence results on the $j$-core.
Proposition 18 (Lekeas 2013, 9-10) Let $C_j$ be the $j$-core of (3) for a symmetric model 9 in (26) given by $p_k(q) = \hat{V} - q_k - \hat{\gamma}\Sigma_{m \neq k}q_m$ and $C_k(q_k) = cq_k$, all $k$.

i) Assume $\hat{\gamma} = 1$ and $j(s) \geq 2(\sqrt{n/s} - 1)$, all $s$. Then, $C_j \neq \emptyset$. ii) Assume $0 < \hat{\gamma} < 1$ and $n \geq 2$. Then, there exists $j^* = j^*(n, \hat{\gamma})$, $0 < j^* \leq 1$ such that for $j(s) > j^*$, all $s$, $C_j \neq \emptyset$. iii) Assume $-1/(n - 1) < \hat{\gamma} < 0$. Then, $C_j \neq \emptyset$.

Thus, by part iii), the $j$-core is always non-empty if goods are complements ($\hat{\gamma} < 0$) and if the complementation parameter is small ($|\hat{\gamma}| < 1/(n - 1)$). If goods are substitutes ($\hat{\gamma} > 0$), by parts i-ii), a non-empty $j$-core requires that outsiders are divided into large number of coalitions. This condition makes it hard to have a non-empty $j$-core because the belief function $j(s)$ is bounded from above by $n - s$. Note that the condition in part ii) is originally stated as $j^* \leq n - s$, all $s$, in Theorem 1 in Lekeas (2013, 9), which has been simplified to $j^* \leq 1$ in above proposition.

Lekeas and Stamatopoulos (2014) study a homogeneous Cournot model with $C_k(q_k) = cq_k$ and $Q = 1 - p^b$, $b > 0$. In this case, the game $\Gamma_f$ in (16) is well defined. They consider a reasonable belief $f(s) = \{f_j(s) | j = 1, ..., n - s\}$ defined by the Sterling number of the second kind (Lekeas and Stamatopoulos 2014, 258), and provide a $f$-core result in linear cases ($b = 1$). However, due to the involved complexity, no $f$-core result is available in nonlinear cases ($b \neq 1$).

Proposition 19 (Lekeas and Stamatopoulos 2014, 262) Let $C_f$ be the $f$-core of (3) for $Q = 1 - p$ and $C_k(q_k) = cq_k$, all $k$, with the above $f$-belief. Then, $C_f \neq \emptyset$ if $n$ is sufficiently large.

Finally, Currarini and Marini (2003) study the leader-follower belief and give a $lf$-core result (called $\lambda$-core in Currarini and Marini (2003, 2015)) in symmetric linear Cournot oligopolies. For each $S \neq N$, let $q_{-S}(q_S) = \{q_j | j \in N \setminus S\}$, $q_j \in ArgMax\{\pi_j(q_j, q_{-j}) | q_j \geq 0\}$, all $j \in N \setminus S$, be the followers’ reaction function, and
$q^*_S \in \text{ArgMax}\{\sum_{i \in S} \pi_i(q_S, q_{-S}(q_S)) | q_S \geq 0\}$ be the leaders’ optimal choices. Then, the leaders’ payoff and the \(lf\)-coalitional game are \(v_{lf}(S) = \sum_{i \in S} \pi_i(q^*_S, q_{-S}(q^*_S))\) and \(\Gamma_{lf} = \{N, v_{lf}(\cdot)\}\).

**Proposition 20** (Currarini and Marini 2003) Let \(C_{lf}\) be the core of \(\Gamma_{lf}\) for a symmetric (34). Then, \(C_{lf} \neq \emptyset\) and equal split is its unique core vector.

It follows from Lemma 4 and Propositions 13 and 20 that \(C_{lf} \subseteq C_\gamma \subseteq C_{\alpha*} \subseteq C = C_\alpha = C_\beta\) hold in symmetric linear Cournot oligopolies.\(^{29}\) It remains to be seen if new non-trivial inclusion results among the above seven core refinements (i.e., \(C_{\alpha*}, C_\gamma, C_e, C_\delta, C_j, C_f\) and \(C_{lf}\)) can be found in future research.

### 3.4 Extensions

This subsection lists seven areas for future research or extensions of the earlier core results:

1) Extend the special cases in Propositions 1 – 20 in the previous two subsections to more general cases of model 9 and then extend these core results in standard Cournot models to the remaining nine models or precisely to models 1 – 5 in (23) and models 6-8 and 10 in (26). This includes the core and its seven refinements in the multi-products or multi-markets in Grossmann (2007), Kao and Menezes (2009), Lapan and Hennessy (2006), Wang and Zhao (2010) and Zhang and Zhang (1996).

2) Extend the reviewed results to mixed oligopolies. Kamaga and Nakamura (2007) provide a core result in a three-firm mixed Cournot oligopoly with linear demand and quadratic costs, but it is not clear what their core is because none of the known \(\alpha\)-, \(\beta\)-, \(\gamma\)- and \(\delta\)-core results are cited in their paper.

\(^{29}\) Currarini and Marini (2004) provide related existence results on the \(lf\)-core and the \(\gamma\)-core, Driessen, Hou and Lardon (2011) also provide a related \(lf\)-core result. See Currarini and Marini (2015, 13-14) for more discussion about the \(lf\)-core.
3) Extend the reviewed results to oligopolies with indivisibility (OI, this simplifies oligopoly market with indivisibility or OMI in Zhao 2000), which are small markets for indivisible or discrete goods (such as superstars in sports and ocean shipping with a small number of large orders) where a one-unit change in demand or supply will have a non-negligible effect on equilibrium. Motivated by Telser’s flight game (Telser 1994, see Example 4 below), Zhao (2000) models a $m$-buyer $n$-seller OI as
\[
OI = \{A, B; C_i(x_i), [0, z_i], p_\alpha\},
\]
where $A = \{1, \ldots, n\}$ is the set of firms or sellers, $B = \{1, \ldots, m\}$ is the set of buyers; $C_i(x_i), x_i \in [0, z_i]$, with $z_i > 0$ as capacity, is the cost function of each firm $i \in A$, and $p_\alpha \geq 0$ is the reservation price of each buyer $\alpha \in B$ for one unit of the homogeneous good. This differs from (30) only in that both $x_i$ and $z_i$ are integers and the inverse demand is replaced by a vector of reservation prices $p \in R^m_+$. A coalitional game $\Gamma_c = \{N, v(\cdot)\}$, $N = A \cup B$, can be defined by computing the profit $v(S)$ for each $S = \{T_A, T_B\} \subseteq \{A, B\}$, with $T_A \subseteq A$ and $T_B \subseteq B$.\textsuperscript{30}

Let a linear cost be given by $C_i(x_i) = b_i$ if $x_i = 0$; $= d_i + c_i x_i$ if $x_i = 1, \ldots, z_i$; and $= \infty$ if $x_i > z_i$, where $d_i$ and $c_i$ are the fixed and marginal costs, $b_i \geq 0$ is the opportunity cost if $b_i < 0$, the sunk cost if $b_i = d_i$, and it makes $d_i$ an avoidable cost if $b_i = 0$. Such OI with linear costs can be defined by a $(4n + m)$-vector $\{b, c, d, z, p\} \in R^n \times R^2_+ \times R^m_+ \times R^m_+$, with $p, z, d, c$ and $b$ as the vectors of reservation prices, capacities, fixed-, marginal- and opportunity-costs, respectively. In OI with only opportunity cost (i.e., $c = d = 0$), (49) is reduced to a $(2n + m)$ vector $\{b, z, p\} \in R^n_+ \times R^m_+ \times R^m_+$. Telser’s flight game is a 2-seller 3-buyer OI with only opportunity cost given here.

**Example 4** (Flight game, Telser 1994) $\{b, z, p\} = \{(85, 150), (2, 3), (70, 60, 55)\}$.

\textsuperscript{30}See Zhao (2000, 184-186) for details. Note that $A$ and $B$ have been switched from those in Zhao (2000) to emphasize that $B$ is the set of buyers.
There are 3 passengers whose reservation prices for a trip are $70, $60 and $55, respectively, and there are 2 private jets (or cabs), one with an opportunity cost $b_1 = $85 and capacity $z_1 = 2$, and the other with $b_2 = 150$ and $z_2 = 3$.

**Proposition 21** (Zhao 2000, 191) Given \( \{b, z, p\} = \{(b_1, b_2), (z_1, z_2), (p_1, p_2, p_3)\} \), with $b_1 < b_2$ and $p_1 \geq p_2 \geq p_3$. Under usual conditions, the core is empty \( \iff 3b_1/2 < b_2 < 3p_3 \).

By $3b_1/2 = 127.5 < b_2 = 150 < 3p_3 = 165$, the core is empty in Example 4. Bejan and Gómez (2009) provide a non-core solution or core extension for such empty core games.


5) Extend the reviewed results to dynamic games as surveyed in Bischi, Lamantia and Radi (2017) and Long (2010).

6) Study the external stability (called comparative statics in economics) of the core such as its upper semi-continuity ($u.s.c$) and lower semi-continuity ($l.s.c$). Applying the $l.s.c$ condition for an optimal set (Zhao 1997) to $mnbp$ in (5) should lead to non-trivial results, which could shed light on studying merger contracts under uncertainty. This should not be confused with internal stability caused by coalitional deviations.

7) Connect the above core results to the huge literature on the noncooperative approach to coalition formation (Bloch 1997, Currarin and Marini (2006, 2015), Ray and Vohra (1997, 2015)). Some of the stable monopolies formed in such studies appear to be a refinement of the core (= $\alpha$-core = $\beta$-core), but this is not totally clear and needs to be verified in future studies, because the known $\alpha$- or $\beta$-core results and connections to the partition function game (3) via the MOG (46) or quasi-hybrid
4 Stable partitions as candidates of non-monopoly solutions

There are no published and only two unpublished studies on the stability of a general non-monopoly partition. The main ideas in the author’s 20-year-old working paper (Zhao 1996) are surprisingly still new. Only the basic idea in this old paper and one result in (Zhao 2013) are reviewed here.

Given a non-monopoly partition $\Delta = \{S_1, S_2, ..., S_h\} \neq \{N\}$ in (30), any notion of its stability must have two basic elements: an unprofitable monopoly merger caused by merging costs, and a payoff vector $\theta = \theta(\Delta) = \{\theta_S| S \in \Delta\} \in R^m_+$ satisfying $\Sigma_{j \in S} \theta_j = \phi_S(\Delta)$ for each $S \in \Delta$, where $\phi_S(\Delta)$ is given in (3) for (30). Thus, this section assumes that for each $\Delta \neq \{N\}$, $\Sigma_{S \in \Delta} \phi_S(\Delta) > v(N) = (\pi^m - mmc)$ with some positive monopoly merging cost $mmc > 0$. The concept of hybrid solution with a distribution rule (HSDR in Zhao 1996 or Zhao 1999a) was introduced to define the other basic element.

Let $D = \{\text{core, equal surplus split, nucleolus, proportional split, Shapley value}\}$ be the set of 5 solutions, which are restricted to (30) and exclude other solutions whose general existences are either unknown or too involved. A distribution rule (DR) for the given $\Delta$ specifies a solution $DR(S) \in D$ for each $S \in \Delta$.

**Definition 5** (Zhao 1996, 1999a) Given a partition $\Delta \neq \{N\}$ and its DR in an oligopoly (30), its hybrid solution with a distribution rule or HSDR is a pair $(q^*, \theta)$ such that $q^*$ is the solution of (20) and for each $S \in \Delta$, $\theta_S$ is its solution.
defined by \( DR(S) \).

Keep in mind that \( \theta_S \) or \( DR(S) \) solves the parametric normal form game

\[
\Gamma_S = \Gamma_S(q^*_S) = \{ S, X_i, \pi_i(q_S, q^*_S) \}
\]

for each \( S \in \Delta \), where \( X_i = [0, z_i] \), \( \pi_i(q_S, q^*_S) = p(\Sigma_{j \in S} q_j + \Sigma_{j \notin S} q^*_j)q_i - C_i(q_i), i \in S \), and \( q^* = \{ q^*_S | S \in \Delta \} \) is the quasi-hybrid solution or postmerger equilibrium in (20). Such \( HSDR \) can be called the merger contracts for \( S \), which extends the monopoly merger contract \( (N, \bar{q}, \theta) \) to a partition contract \( (\Delta, q^*, \theta) \), specifying that each merger \( S \in \Delta \) distribute its profits by a solution \( DR(S) \in D \).

Now, given a partition contract \( (\Delta, q^*, \theta) \) for \( \Delta \neq \{N\} \), consider the possible deviation by a coalition \( S \notin \Delta, S \neq N \). \( S \) has incentives to move to a new partition \( \Delta' = \{ S, T_1, ..., T_m \} \in \Pi(S) \) in (10) if its payoff at the new partition is higher than the sum of its members’ current payoffs or if \( \phi_S(\Delta') > \Sigma_{j \in S} \theta_j \), with \( \phi_S(\Delta') \) given in (3) for (30). A stable contract \( (\Delta, q^*, \theta) \) should rule out all such deviations.

Note that most stable partitions identified in the noncooperative approach require symmetry, and some of them (such as the equilibrium-binding agreement in Ray and Vohra 1997) only rule out a small set of possible coalitional deviations, such stable partitions are thus very weak and are not really stable. This is the reason why they have been excluded here as candidates of non-monopoly solutions.

For simplicity, only the \( \gamma^- \), \( \delta^- \), \( \alpha^+ \)- and \( \epsilon^- \)-deviations are evaluated here. The new partitions \( \Delta_{\alpha^+} = \Delta_{\alpha^+}(S) = \{ S, T_1^{\alpha^+}, ..., T_{m(\alpha^+)}^{\alpha^+} \} \) and \( \Delta_{\epsilon} = \Delta_{\epsilon}(S) = \{ S, T_1^{\epsilon}, ..., T_{m(\epsilon)}^{\epsilon} \} \) are the same as in (13-14), due to their independency of the current \( \Delta \). The new partitions \( \Delta_{\gamma^-} = \Delta_{\gamma^-}(S, \Delta) \) and \( \Delta_{\delta^-} = \Delta_{\delta^-}(S, \Delta) \) under \( \gamma^- \) and \( \delta^- \)-beliefs are given by

\[
\Delta_{\gamma^-} = \Delta_{\gamma^-}(S, \Delta) = \{ S, T_1^{\gamma^-}, ..., T_{m(\gamma^-)}^{\gamma^-} \} \in \Pi(S), \quad \text{and} \quad \Delta_{\delta^-} = \Delta_{\delta^-}(S, \Delta) = \{ S, T_1^{\delta^-}, ..., T_{m(\delta^-)}^{\delta^-} \} \in \Pi(S),
\]
where for each \(i = 1, \ldots, m(\gamma), T^\gamma_i = T\) for each \(T \in \Delta\) with \(S \cap T = \emptyset, = \{j\}\) for each \(j \in T \setminus S\) and each \(T \in \Delta\) with \(S \cap T \neq \emptyset\); and for each \(i = 1, \ldots, m(\delta), T^\delta_i = T \setminus S = \{j | j \in T, j \notin S\}\) for each \(T \in \Delta\). As an example, for \(\Delta = \{1, 2345, 67\}\) and \(S = \{1, 2\}\), one has \(\Delta_\gamma = \{12, 3, 4, 5, 67\}\) and \(\Delta_\delta = \{12, 345, 67\}\).

**Definition 6** (Zhao 1996) A partition contract \((\Delta, q^*, \theta)\) for \(\Delta \neq \{N\}\) or \(\Delta\) with \(\theta(\Delta)\) is \(\gamma\)-stable (\(\delta\)-, \(\alpha^*\)- and \(e\)-stable) if for all \(S \notin \Delta\), \(\Sigma_{j \in S} \theta_j \geq \phi_S(\Delta_\gamma)\) \((\geq \phi_S(\Delta_\delta), \geq \phi_S(\Delta_{\alpha^*})\) and \(\geq \phi_S(\Delta_e))\), where \(\Delta_{\alpha^*}\) and \(\Delta_e\) are given in (13-14), \(\Delta_\gamma\) and \(\Delta_\delta\) are given in (51-52).

Let the \(mnbp\) under the above four notions of stability be given by

\[
\text{mnbp}_k = \{\text{Min} \sum_{i=1}^n x_i | x \geq 0, \Sigma_{i \in S} x_i \geq \phi_S(\Delta_k), \text{all } S \notin \Delta, S \neq N\} \tag{53}
\]

for \(k = \gamma, \delta, \alpha^*\) and \(e\), and let the optimal set or the set of minimal solutions in (53) be given, respectively, by

\[
Y_\gamma(\Delta), Y_\delta(\Delta), Y_{\alpha^*}(\Delta)\text{ and } Y_e(\Delta). \tag{54}
\]

Then, the stability of \(\Delta \neq \{N\}\) with \(\theta(\Delta)\) is fully characterized by (54) or (53).

**Proposition 22** (Zhao 1996) Given a partition contract \((\Delta, q^*, \theta)\) for \(\Delta \neq \{N\}\) in (30), assume \(v(N) = (\pi^m - \text{mmc}) < \Sigma_{i=1}^n \theta_i\). Then, for \(k = \gamma, \delta, \alpha^*\) and \(e\), \(\Delta\) with \(\theta\) is \(k\)-stable \(\iff \theta \in Y_k(\Delta)_+,\) where \(Y_k(\Delta)_+\) is given in (54) and \(Y_k(\Delta)_+ = \{x + y | x \in Y_k(\Delta), y \in R^n_+\}\).

If players are allowed to freely redistribute \(\theta\) in \((\Delta, q^*, \theta)\) among the \(n\) players, the above conclusions can be simplified as \(\Delta \neq \{N\}\) is \(k\)-stable if and only if for \(k = \gamma, \delta, \alpha^*\) and \(e\), \(\Sigma \theta_i = \Sigma_{S \in \Delta} \phi_S(\Delta) \geq \text{mnbp}_k\), which is given in (53).

The next proposition concludes this survey with a \(\delta\)-stable non-monopoly partition in a three-firm asymmetric linear Cournot oligopoly by applying Proposition 22.
Given \((a, c, z) \in R^7_+\) and \(\Delta_1 = \{1, 23\}, \Delta_2 = \{2, 13\}, \Delta_3 = \{3, 12\}\), one can verify that their \(\gamma\)-, \(\delta\)- and \(\alpha^*\)-stabilities are identical, so there is no need to make such distinction here. The outsider’s or the single firm’s profit at each \(\Delta_k\) \((k = 1, 2, 3)\) is

\[
\begin{align*}
\phi_1(\Delta_1) &= (a - c_1)^2(1 + \varepsilon_2)^2/9, \\
\phi_2(\Delta_2) &= (a - c_1)^2(1 - 2\varepsilon_2)^2/9, \\
\phi_3(\Delta_3) &= (a - c_1)^2(1 - 2\varepsilon_3)^2/9,
\end{align*}
\]  

(55)

where \(\varepsilon_2\) and \(\varepsilon_3\) are given in (47). The merger’s gain in each \(\Delta_k\) is:

\[
\begin{align*}
d_{23} &= \phi_{23}(\Delta_1) - (\pi_2^0 + \pi_3^0), \\
d_{13} &= \phi_{13}(\Delta_2) - (\pi_1^0 + \pi_3^0), \\
d_{12} &= \phi_{12}(\Delta_3) - (\pi_1^0 + \pi_2^0),
\end{align*}
\]  

(56)

where \(\pi_i^0\) is \(i\)’s premerger profit, and \(\phi_S(\Delta)\) is the postmerger profit in (3) for (34). For \(S = 12, 13,\) and \(23\), let the efficient member’s share of the above gains be \(t \in [0, 1]\). Then, the three-dimensional payoff vector \(\theta(\Delta_k) = \theta(t) \in R^3_+\) for \(k = 1, 2, 3\), respectively, is

\[
\begin{align*}
\text{for } \Delta_1, \theta_1 &= \phi_1(\Delta_1), \theta_2 = \pi_2^0 + td_{23}, \theta_3 = \pi_3^0 + (1 - t)d_{23}, \\
\text{for } \Delta_2, \theta_1 &= \pi_1^0 + td_{13}, \theta_2 = \phi_2(\Delta_2), \theta_3 = \pi_3^0 + (1 - t)d_{13}, \text{ and} \\
\text{for } \Delta_3, \theta_1 &= \pi_1^0 + td_{12}, \theta_2 = \pi_2^0 + (1 - t)d_{12}, \theta_3 = \phi_3(\Delta_3).
\end{align*}
\]  

(57)

**Proposition 23** (Zhao 2013, 16) *Given \((a, c, z) \in R^7_+\), suppose \(\Sigma_{i=1}^3 \theta_i > (\pi^m - mmc)\), \(d_S > 0\) for \(S = 12, 13\) and \(23\), and assume parts ii-iii) of A0.1. Then, the following three claims hold:*
i) $\Delta_1$ with $\theta(t)$ is stable (i.e., $\delta$- or $\gamma$- or $\alpha^*$- stable) $\iff \varepsilon_3 \leq \mu_1(\varepsilon_2, t);\varepsilon_3 \leq \mu_1(\varepsilon_2, t)$ if $0 \leq \varepsilon_2 \leq 1/11$, and $\varepsilon_3 > \mu_1(\varepsilon_2, t)$ if $113/316 < \varepsilon_2 \leq 1/2$.

ii) $\Delta_2$ with $\theta(t)$ is stable $\iff \varepsilon_3 \leq \mu_2(\varepsilon_2, t); \varepsilon_3 \leq \mu_2(\varepsilon_2, t)$ if $0 \leq \varepsilon_2 \leq 1/11$, and $\varepsilon_3 > \mu_2(\varepsilon_2, t)$ if $\varepsilon_2 < 1/2$, where $\varepsilon_2(t) = (2t - 9)/[14(2t - 3)]$.

iii) $\Delta_3$ with $\theta(t)$ is stable $\iff \varepsilon_2 \leq \mu_3(\varepsilon_3, t)$, which holds if $0 \leq \varepsilon_3 \leq 3/14$.

The details of $\mu_1(\varepsilon_2, t)$ and $\mu_2(\varepsilon_2, t)$ (note $\mu_3(\varepsilon_3, t) = \mu_2(\varepsilon_3, t)$) can be found in (A27-A29) in Zhao (2013). Although such results appear to be technical, they have intuitive interpretations. Consider, for example, part i) or the stability of $\Delta_1 = \{1, 23\}$. Observe first that $\Delta_0 = \{1, 2, 3\}$ and $\Delta_m = \{123\}$ are ruled out by the assumptions and $\Delta_2 = \{2, 13\}$ has the same postmerger profits of $\Delta_1$; thus, one only needs to evaluate the deviation by $S = 12$ in $\Delta_3$. Because a larger share $t$ by firm 2 or a smaller $\varepsilon_3$ makes the merger of $S = 12$ less profitable, $\Delta_1$ with $\theta(t)$ will be stable with a smaller $\varepsilon_3$ or a larger $t$ (i.e., $\varepsilon_3 \leq \mu_2(\varepsilon_2, t)$, which is increasing in $t$).

5 Empirical studies of the core

Early empirical studies of the core include Bittlingmayer (1982), Sjostrom (1989), Pirrong (1992), and McWilliams and Keith (1994). Reading these papers, it is clear that the authors had an intuitive understanding of the core for which there was no precise model and that their intuition was based on the empty-core examples in their and other early studies such as Faulhaber (1975), Shapley and Shubik (1969) and Telser (1978, 1994). They understood that the core theory assumes $A3$ and that the empty-core was caused by economies of scale in Addyston Pipe (Bittlingmayer 1982), or by low demand plus indivisible supply or avoidable cost in Ocean Shipping Conferences (Pirrong 1992, Sjostrom 1989) and Trust Industries (McWilliams and
Keith 1994); they equated *empty-core* to *market failure* or *ruinous competition*.

The core in these studies involves a small number of sellers and buyers (similar to the oligopoly with indivisibility in (49)), so it is different from the core in oligopolies as defined earlier. The documented evidences on the sellers’ arrangements were interpreted as a solution for the empty-core to avoid cutthroat competition. However, these same evidences on sellers’ arrangements might be interpreted alternatively as supports for a non-empty core in games involving only the sellers. The author believes that a non-empty core of the sellers can be established by revisiting the evidences of Joint Traffic in Addyston Pipe (Bittlingmayer 1982), prices and quotas in Ocean Shipping (Pirrong 1992, Sjostrom 1989) and share allocations in Trusts (McWilliams and Keith 1994).

Recent applications of the core in Propositions 2-5 include liner shipping alliances (Shi and Voss 2011, Yang, Liu and Shi 2011), insurance (Stoyanova and Gruendl 2014) and sugar monopoly (Zhao 2009b). Stoyanova and Gruendl (2014) study the EU legislation called Solvency II, which replaced thirteen old EU insurance directives on January 1, 2016. Their conclusion is that Solvency II will reduce merging costs and drive more mergers and acquisitions in the EU insurance industry.

Zhao (2009b) applies Proposition 5 to the 1887-1914 sugar monopoly (Eichner 1969, Genesove and Mullin 1998, and Wang 2008), which replaced the 1882-1887 Sugar Trust (McWilliams and Keith 1994). The monopoly consolidated eighteen refineries in 1887, with an excess capacity rate of about 20%; it dissolved into twelve refineries in 1914, with a near full capacity. Using $n = 18$, $\tau = 0.20$, and the estimated linear model in Genesove and Mullin (1998), the estimated monopoly merging costs are at most 35% of pre-merger total profits at its formation in 1887. Using $n = 12$ and $\tau = 0$, the estimated monopoly’s organizational costs (i.e., the costs of keeping monopoly and avoiding dissolution) are at least 252% of post-dissolution total profit.
at its dissolution in 1914. These results provide a new understanding about the formation and dissolution of the sugar monopoly: it was formed in 1887 because its merging cost was sufficiently low, and it was dissolved in 1914 because enforcing the Sherman act increased its organizational costs to a level that was too high to be operational. Zhao (2009b) also reports similar results using linear demand and quadratic costs.

6 Conclusion and future study

The process from early division of labor or specialization to modern industrial organization is long and dates back at least 170,000 years.\textsuperscript{31} Such a long process in human history is powered and pulled forward by its two indisputable driving horses or driving wheels called \textit{competition} and \textit{cooperation}.

The previous literature in industrial organization has largely focused on \textit{competition} or the application of noncooperative game theory, with the exception of a small group of scholars whose works on \textit{industrial cooperation} have been reviewed with some details in this survey. Readers are encouraged to extend the surveyed results to more general and more realistic models. In addition to the seven extensions listed in sub-section 3.4, applied scholars are encouraged to extend the few empirical core studies to all industries or sectors with merger activities or joint ventures, and theoretically minded scholars are encouraged to extend the existing core results to non-monopoly partitions.

\textsuperscript{31}One of the earliest evidences of specialization is a workshop or small factory for making stone tools, recently confirmed to be at least 170,000 years old, at the Dingcun Site (Ding village in Xiangfen) in Shanxi, China. See Wang (2015). The entry of Dingcun on both Wiki and UNESCO World Heritage Tentative List is about ancient civilian residential houses in the same village.
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