The $\delta$-core and Stable Partitions in Asymmetric Oligopolies

By Jingang Zhao*

February 2009

Department of Economics
University of Saskatchewan
9 Campus Drive
Saskatoon, Saskatchewan
CANADA S7N 5A5
Tel: (306) 966-5217
Fax: (306) 966-5232
Email: j.zhao@usask.ca

Abstract: This paper solves merger formation problem in Cournot oligopolies by a cooperative and computational approach. In three-firm linear oligopolies, monopoly will be formed (or be stable) if its merging cost is sufficiently low and cost differentials are sufficiently large. When monopoly is unprofitable due to high merging costs, a profitable two-member merger will be formed if its efficient member receives a sufficiently large share of the merger’s gain and cost differentials are sufficiently small. In $n$-firm linear oligopolies, monopoly will be formed if number of firms and merging costs both are sufficiently small and cost differentials are sufficiently large.

JEL Classification Number: C62, C71, C72, D43, L10

Keywords: Core, $\delta$-core, Cournot equilibrium, hybrid equilibrium, merger analysis, stable partition

* This paper replaces an earlier working paper titled “A Stable Market Structure as the Solution for Cournot Oligopolies” (2002). I would like to thank Andreas Blume, Donald Smythe, and seminar participants at American U., Australian National U., Iowa State U., U. Kansas, U. Melbourne, U. Missouri, U. Saskatchewan, and 2001 NASMES at U. Maryland for comments on the earlier draft. All errors, of course, are my own.
1. Introduction

Consider the four possible mergers in an asymmetric oligopoly with three firms: 123; 12, 13 and 23. Which one, if any, will be formed and be free of subsequent breakups or new mergers? Although the question seems fairly simple, it is one of the most fundamental questions in both game theory and industrial organization that has not been thoroughly studied prior to this study. This paper provides a complete answer for linear Cournot oligopolies by studying stable partition or stable market structure, which is defined as a set of simultaneous mergers that are free of profitable subsequent mergers or breakups.

Briefly answering the question (in order), monopoly will be formed if monopoly merging cost is low and the two inefficient firms are sufficiently small; and each of the two-member mergers will be formed if i) it is profitable; ii) monopoly is unprofitable (due to high merging cost); iii) efficient member’s share of the merger’s gain is sufficiently large; and iv) the inefficient member is not too small. The paper provides complete characterizations for each of the five partitions to be stable over the space of three firm linear oligopolies.

Next, it provides complete characterizations for stable monopoly in two classes of \( n \)-firm linear oligopolies. Finally, the paper provides computational characterizations for stable partitions in general \( n \)-firm oligopolies. Our cooperative and computational approach has two advantages over the previous literature. First, it allows us to investigate how asymmetry such as cost differentials affect a stable partition, while most previous works, in both cooperative (e.g., Yong [2004], Funaki and Yamato [1999] and Rajan [1989]) and non-cooperative (see Yong [2004] and Ray and Vohra [1999] for short surveys) approaches, have assumed that players have symmetric or identical payoff functions. Second, it allows us to study the effect
of merging cost (the cost of coalition formation)\(^1\) on stable partitions, which has not been studied in the previous literature.

The rest of the paper is organized as follows. Section 2 describes the model and merger contacts; section 3 defines the \(\delta\)-core and stable partitions; and section 4 characterizes stability in three firm linear oligopolies. Section 5 extends the results to \(n\)-firm markets, section 6 concludes, and the appendix provides proofs.

2. Description of the model and merger contracts

A linear Cournot oligopoly for a homogeneous good is given by a linear inverse demand \(p(\Sigma x_j) = a-\Sigma x_j\) and \(n\) linear cost functions: \(C_i(x_i) = c_i x_i, 0 \leq q_i \leq z_i, i = 1, \ldots, n\), which can be defined by a \((2n+1)\)-vector \((a, c, z) \in \mathbb{R}^{2n+1}_{++}\), with \(a > 0\) as the intercept of inverse demand, \(c = (c_1, \ldots, c_n) >> 0\) as vector of marginal costs, and \(z = (z_1, \ldots, z_n) >> 0\) as vector of capacities.

Without loss of generality, assume \(c_1 \leq \ldots \leq c_n\), so firm 1 is most efficient and firm \(n\) is least efficient. Alternatively, the above model is equivalent to the following normal form game:

\[
\Gamma = \{N, Z_i, \pi_i\}, \tag{1}
\]

where \(N = \{1, 2, \ldots, n\}\) is the set of firms; for each firm \(i \in N\), \(Z_i = [0, z_i]\) is its production set bounded by its capacity \(z_i > 0\), and \(\pi_i(x) = p(\Sigma x_j)x_i - C_i(x_i) = (a-\Sigma x_j-c_i)x_i\) is its profit function.

We assume that each merger generates a weak synergy denoted as the following A0:

\[\text{A0 (Assumption 0)}: (i)\text{ For each merger } S \subseteq N, \text{ its capacity and cost function are:}
\]

\[
z_S = \Sigma_{j \in S} z_j, C_S(q) = c_S q, q \leq z_S, \text{ where } c_S = \min \{c_j | j \in S\}; \text{ and}
\]

\[(ii)\text{ at any equilibrium, the optimal supply by each } S \subseteq N \text{ is an interior solution.}\]
Under A0, a merger removes inefficient members and raises the efficient member’s capacity to $z_S$. For each non-monopoly merger $S \neq N$, its guaranteed (or worst) profit is:

$$v(S) = \max_{x_S} \min_{y_S} \sum_{i \in S} \pi_i(x_S, y_S) = \min_{y_S} \max_{x_S} \sum_{i \in S} \pi_i(x_S, y_S),$$

which defines a coalitional game

$$\Gamma_c = \{N, v\},$$

(3)

where the Min in $v(S)$ is taken over $Z_S = \prod_{j \notin S} Z_j$, the Max over $\{x_S \in \mathbb{R}^S | \sum_{j \in S} x_j \leq z_S\}$, $(x_S, y_S) = w$ is a vector with $w_i = x_i$ if $i \in S$, $= y_i$ if $i \notin S$, and $v(N) = \bar{\pi}$ is the monopoly profits. The core of monopoly merger or the core of (1) is the core of (3), which is denoted as

$$\text{Core}(\Gamma) = \{\lambda \in \mathbb{R}^n | \sum_{i \in S} \lambda_i \geq v(S), \text{ all } S \neq N\}.$$  

(4)

In words, a profit vector is in the monopoly’s core if it splits the monopoly profits in such a way that it is unblocked by each non-monopoly merger.

Let $\bar{x}$ denote monopoly supply. Define a monopoly merger contract as a pair $(\bar{x}; \lambda)$ of the monopoly’s supply and a split of its profits. Then, a precondition for each monopoly merger contract $(\bar{x}; \lambda)$ is that $\lambda \in \text{Core}(\Gamma)$. Zhao (2001) showed the equivalence between $\text{Core}(\Gamma) \neq \emptyset$ and $v(N) \geq MNBP$, where $MNBP$ is the minimum no-blocking payoff given by

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2 In more general situations, one would have

$$v_\alpha(S) = \max_{x_S} \min_{y_S} \sum_{i \in S} \pi_i(x_S, y_S) < v_\beta(S) = \min_{y_S} \max_{x_S} \sum_{i \in S} \pi_i(x_S, y_S),$$

which implies $\beta$-core $\subset \alpha$-core (Aumann [1959], Scarf [1971]). Since $\alpha$-core $= \beta$-core holds in an oligopoly (Zhao, 1999), there is no need to make the $\alpha$- and $\beta$-distinction here and one can simply use the term core.

3 The non-empty core and profitability preconditions are independent of each other. As an example, let $n = 3$, $(a; c; z) = (6; 0.5, 0.5, 0.5; 2, 2, 2)$, and merging costs be $MC_N = 2$, $MC_N = 0$, $S \neq N$. Then, $\pi_i = 1.89$, $\pi_m = 7.56$, $MNBP = 4.59 < v(N) = (\pi_m - MC_N) = 5.56 < \sum \pi_i = 5.67$, so monopoly has a non-empty core and is unprofitable.
\[ MNBP(T) = \begin{cases} 
\min \sum x_i \\
\text{subject to } x \in \mathbb{R}^n; \Sigma_{i \in S} x_i \geq v(S) \text{ for all } S \neq N. 
\end{cases} \tag{5} \]

As readers will see, one major task in this study is to obtain the precise expressions of \( MNBP \) in terms of the parameter vector \((a, c, z)\).

Bain (1959) defined market structure as “the number and size distribution of firms”. In contrast, we define market structure as “the number and size distribution of mergers” or a set of simultaneous mergers, which is a partition of the firms \( \Delta = \{ S_1, \ldots, S_J \} \) (i.e., \( \cup S_j = N \), \( S_j \cap S_i = \emptyset \) for all \( i \neq j \)). A market structure \( \Delta \) assumes that binding contracts are, at the time of consideration, available for each merger \( S \in \Delta \) and unavailable for all other mergers \( T \not\in \Delta \).

Given a partition \( \Delta \), let \( \tilde{x}(\Delta) = \{ \tilde{x}_S(\Delta) \mid S \in \Delta \} \) and \( \tilde{\pi}(\Delta) = \{ \tilde{\pi}_S(\Delta) \mid S \in \Delta \} \) be the unique post-merger supply and profit vectors.\(^4\) Then, a merger contract for \( \Delta \) is a list of pairs \( \{ \tilde{x}_S(\Delta); \tilde{\lambda}_S(\Delta) \} \) of post-merger supply and split of post-merger profits (i.e., \( \tilde{\lambda}_S \geq 0 \), \( \Sigma_{j \in S} \tilde{\lambda}_j = \tilde{\pi}_S(\Delta) \)) for each merger \( S \in \Delta \). The non-empty core precondition requires that each \( \tilde{\lambda}_S \) be in the core of \( S \), or equivalently, in the core of the following normal form game:

\[ \Gamma_S(\tilde{x}, \tilde{\lambda}) = \{ S, Z_s, \pi(x_S, x_{\sim S}) \}, \tag{6} \]

where \( \pi(x_S, \tilde{x}) = (a - \sum_{j \in S} x_j - \sum_{j \not\in S} \tilde{x}_j - c_i) x_i, i \in S \), are parameterized by the fixed \( \tilde{x} \).

Note that the core for each \( S \in \Delta \) is only defined for the fixed outsiders’ supply \( \tilde{x}_S(\Delta) \). Otherwise, the problem of dividing its post-merger profits would not exist. At the post-merger equilibrium, each merger \( S \in \Delta \) splits its profits within its core, so the merger contract \((\tilde{x}(\Delta), \tilde{\lambda}(\Delta)) = \{ (\tilde{x}_S, \tilde{\lambda}_S) \mid S \in \Delta \} \) specifies a hybrid equilibrium (Zhao, 1992).

\[^4\] \( \tilde{x}(\Delta) \) satisfies: for each \( S \in \Delta \), \( 0 \leq \Sigma_{j \in S} \tilde{x}_j \leq z_S \quad \tilde{x}_S(\Delta) = \Sigma_{j \in S} \pi_j(\tilde{x}_S, \tilde{x}_j) \geq \pi_S(y_S; \tilde{x}_S) \) for all \( 0 \leq \Sigma_{j \in S} y_j \leq z_S \).
As shown in Figure 1, the spectrum of hybrid equilibria includes the core of monopoly merger (for $\Delta_m = \{N\}$) and Cournot equilibrium (for $\Delta_0 = \{(1), \ldots, (n)\}$) as its two end points, which will be endogenized as a stable partition in the next three sections.

**Figure 1.** The spectrum of hybrid equilibria, where $k$ ($1 \leq k \leq n$) is the number of coalitions in $\Delta$.

### 3. $\mathcal{D}$-core as a special case of stable partitions

Let $\Pi$ be the set of partitions of $N$. For each $\Delta \in \Pi$, its unique post-merger profits $\tilde{\pi}(\Delta)$ convert (1) into a partition function game (Thrall and Lucas, 1963):

$$\Gamma_{PF} = \{N, \phi\},$$  \hspace{1cm} (7)

which specifies a joint profit $\phi(T, \Delta) = \tilde{\pi}(\Delta)$ for each merger $T \in \Delta$ and each partition $\Delta \in \Pi$.

Given $\Delta$ and its merger contract $(\tilde{\pi}(\Delta), \lambda(\Delta)) = \{(\tilde{\pi}_T, \lambda_T)\}_{T \in \Delta}$, consider the deviation by (or possible formation of) a different merger $S \not\in \Delta$ (this implicitly assumes that binding contracts now are available to $S$ and all of its proper sub-coalitions). Let

$$\Pi(S) = \{\mathcal{B} \in \Pi \mid \mathcal{B} = \{S, T_p, \ldots, T_m\}\},$$  \hspace{1cm} (8)

denote the set of partitions of which $S$ is a member. $S$ has incentives to move to $\mathcal{B} \in \Pi(S)$ if $\phi(S, \mathcal{B}) = \tilde{\pi}_S(\mathcal{B}) > \sum_{j \in \mathcal{B}} \phi_j(\Delta)$. Hence, "whether $S \not\in \Delta$ will deviate from $\Delta$" or "whether a new
merger $S \not\in \Delta$ will be formed” depends on two factors: i) its members’ current profits on the contract $\lambda(\Delta)$, and ii) its joint profits at the new partition $B \in \Pi(S)$.

In reality, merger contracts, although binding, usually have clauses allowing members to break up the deal under penalty and specifying how the remaining members will react to a breakup. Two common reactions to a breakup (Hart and Kurz, 1983) are: i) remaining members breakup into singletons, and ii) remaining members remain loyal to each other and stay together as a smaller coalition, which are formally given as below:

$$B_\delta(S, \Delta) = \{S, T_1^\delta, \ldots, T_m^\delta\} \in \Pi(S) \text{ (for loyal belief), and}$$

$$B_\gamma(S, \Delta) = \{S, T_1^\gamma, \ldots, T_m^\gamma\} \in \Pi(S) \text{ (for breakup belief),}$$

where for $j = 1, \ldots, m(\delta)$, $T_j^\delta = T/S = \{i \mid i \in T, i \not\in S\}$ for some $T \in \Delta$; and for $i = 1, \ldots, m(\gamma)$, $T_i^\gamma = T$ for some $T \in \Delta$ with $S \cap T = \emptyset$.

As an example, for $\Delta = \{1, (2,3,4,5)\}$ and $S = (1,2)$, one has: $B_\delta(S, \Delta) = \{(1,2), 3, 4, 5\}$, $B_\gamma(S, \Delta) = \{(1,2), (3,4,5)\}$. In addition to the above two beliefs, another well known belief is the worst partition (among $\Pi(S)$) based on cautious belief as given below:

$$B_{\alpha}(S, \Delta) \equiv B_{\alpha}(S) = \{S, T_1^\alpha, \ldots, T_m^\alpha\} \in \Pi(S) \text{ (for cautious belief),}$$

which is the solution of $\text{Min}\{\phi(S, B) \mid B \in \Pi(S)\}$. Note that the above worst partition for $S$ is independent of the current partition $\Delta$. Definition 1 below defines stable partitions.

**Definition 1:** A partition $\Delta$ is $\alpha$- ($\gamma$-; $\delta$-) stable or stable under the cautious (breakup; loyal) belief if it has a merger contract $(\kappa(\Delta), \lambda(\Delta))$ such that for all $S \not\in \Delta$, $\Sigma_{j \in S} \lambda_j(\Delta) \geq \Sigma_{j \in S} \kappa_j(\Delta)$.  

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5 Yong (2004) introduces the efficient-belief, under which outsiders choose the efficient one among all partitions of $N\backslash S$. Let $v_\gamma(S)$, $v_\delta(S)$, $v_e(S)$ denote the value of $S$ under $\delta$, $\gamma$- and $e$- beliefs; and $X_\gamma$, $X_\delta$, and $X_e$ the associated payoffs. Then, $v_\delta(S) \geq v_\gamma(S)$, $v_\gamma(S) \geq v_e(S) > v(S)$ and $v_e(S) \geq v_\delta(S)$ lead to $\{X_\gamma \cup X_\delta\} \subset X_e \subset \text{Core}$.  

7
\(\phi(S, B_\alpha(S)) \geq \phi(S, B_\beta(S, \Delta)) \geq \phi(S, B_\delta(S, \Delta))\), where \(\phi, B_\alpha, B_\beta,\) and \(B_\delta\) are given in (7)-(11).

To put it differently, mergers in \(\Delta\) will be formed in the \(\alpha\)- (\(\gamma\), \(\delta\)) fashion if no other merger \(S\) could make more profits by moving to \(B_\alpha(S)\) (\(B_\beta(S, \Delta), B_\delta(S, \Delta)\)). Because all profitable breakups and mergers are ruled out, a stable partition or a stable market structure is a candidate for the solution of oligopoly markets. For each market structure \(\Delta\), let

\[X_\alpha(\Delta), X_\gamma(\Delta), \text{ and } X_\delta(\Delta)\]

(12) denote respectively the sets of its \(\alpha\), \(\gamma\) and \(\delta\) stable profit vectors (or precisely sets of profit vectors in its \(\alpha\), \(\gamma\) and \(\delta\) stable merger contacts). Two remarks about (12) are in order.

First, we refer the above \(X_\gamma(\Delta_m)\) and \(X_\delta(\Delta_m)\) as the monopoly’s \(\gamma\) and \(\delta\)-core (or the \(\gamma\) and \(\delta\)-core of (1)), where \(\Delta_m = \{N\}\) is the grand coalition or the coarsest partition. If \(\Delta\) is not the grand coalition, it will be improper to apply the term “core” to the payoffs in (12) because core is reserved for the grand coalition. Hence, for \(\Delta \neq \Delta_m\), the payoffs in (12) will be referred as the \(\alpha\), \(\gamma\) and \(\delta\)-stable payoffs of \(\Delta\), respectively. Second, it will also be improper to refer \(X_\alpha(\Delta_m)\) as the \(\alpha\)-core because the term \(\alpha\)-core has been used already (i.e., the core in (4) is the \(\alpha\)-core). Therefore, we refer \(X_\alpha(\Delta_m)\) as the monopoly’s \(\alpha\)-stable payoff. The difference between \(X_\alpha(\Delta_m)\) and the \(\alpha\)-core is that there exist strategic interactions between each \(S\) and its outsiders \(N\backslash S\) in \(X_\alpha(\Delta_m)\), while there are no such interactions in the \(\alpha\)-core.

Proposition 1 below summarizes the relationship among the above stability concepts.

**Proposition 1:** Given a market structure \(\Delta\) of (1), let \(\text{Core}(\Gamma), X_\alpha(\Delta), X_\gamma(\Delta),\) and \(X_\delta(\Delta)\) be given in (4) and (12), respectively. Then, the following two claims hold:

(i) \(X_\delta(\Delta) \subset X_\gamma(\Delta) \subset X_\alpha(\Delta) \subset \text{Core}(\Gamma);\) and

(ii) \(X_\delta(\Delta_m) \subset X_\gamma(\Delta_m) = X_\alpha(\Delta_m) \subset \text{Core}(\Gamma).\)
By the proposition, $\delta$-stability is stronger than $\gamma$-stability, which is stronger than the $\alpha$-stability, and they are all refinements of the core, whose existence has been reported in Norde et al (2002) and Zhao (2009). Part (ii) shows that the monopoly’s $\alpha$- and $\gamma$-stabilities are identical, because each $S$ has the lowest profits when the outsiders are singletons. Although one might favor one version of stability over another, only future empirical evidences could determine which version is the most appropriate one.

4. Stable market structures with three asymmetric firms

This section characterizes the stability for each of the five partitions of (1) with $n = 3$, which is given by $(a; c; z) = (a; c_1, c_2, c_3; z_1, z_2, z_3) \in \mathbb{R}_+^7(c_1 \leq c_2 \leq c_3)$. It is useful to begin with three observations. First, there is no need to make the $\alpha$ and $\gamma$ distinction for a stable monopoly because its $\alpha$- and $\gamma$-stable sets are identical.

Second, although the three-firm model seems to be restrictive at first glance, evaluating its stability in seven dimensional space is extremely complicated and challenging. Fortunately, we are able to simplify the problem from seven dimensions to two dimensions by assuming A0 and by introducing two intermediate variables:

$$\varepsilon_2 = (c_2-c_1)/(a-c_1) \text{ and } \varepsilon_3 = (c_3-c_1)/(a-c_1),$$

(13)

which represent firm 1’s relative cost advantages over firms 2 and 3. The usual assumptions imply $\varepsilon_2 \leq \varepsilon_3 \leq 0.5$, so all possible cases are covered by the area above the 45$^\circ$ line (see figures 2-4). A large $\varepsilon_2$ represents large cost differentials (or the two inefficient firms are small), and a small $\varepsilon_3$ represents small cost differentials (or the most inefficient firm is not too small).

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6 See Bejan and Gómez (2009), Yong (2004), Funaki and Yamato (1999), and Rajan (1989) for related core refinements.
Even with such simplification, it would have been impossible to carry out this study without mathematical software (see (21) or Figure 5 or the long proof of Lemma 1 for a taste of its complexity). In deed, the involved polynomial equations are solved by Scientific WorkPlace, and all analytical predictions are confirmed separately by numerical examples using Excel.\footnote{The Excel program is available from the author to readers for their classroom use or future research.}

Third, our model is also sufficiently general to analyze the effects of cost differentials, internal allocation and merging costs on stable partitions, which is a major advantage over symmetric models whose predictions will generally collapse in the presence of asymmetry (see Stamatopoulos and Tauman [2009] for an example of such collapse).

The monopoly’s stability and optimality are characterized by comparing $\varepsilon_3$ against the following two functions of $\varepsilon_2$, respectively:

$$
\omega_1(\varepsilon_2) = \frac{2-\sqrt{1+8\varepsilon_2-20\varepsilon_2^2}}{4}, \text{ and }
$$

$$
\omega_2 = \omega_2(\varepsilon_2) = \frac{7+31\varepsilon_2}{69},
$$

Here, the optimal partition $\Delta^*$ is optimal in the sense of second best, as it has the maximal welfare $W^*$ (= total profits + consumer surplus) among the five partitions.

**Proposition 2:** Under $A_0$, the following two claims hold: (i) $\Delta_m = \{123\}$ is always $\alpha$-stable; (ii) $\Delta_m$ is $\delta$-stable $\Leftrightarrow \varepsilon_3 \geq \omega_1(\varepsilon_2)$, and $\varepsilon_3 \geq \omega_1(\varepsilon_2)$ always holds if $\varepsilon_2 \in [1/6, 1/2]$.

By Proposition 2, monopoly will be formed in $\delta$-fashion if firms 2 and 3 have very small market shares (e.g., $\varepsilon_3 \geq \varepsilon_2 \geq 1/6$), and it will be formed in $\alpha$-fashion but not $\delta$-fashion if firms 2 and 3 have similar and sizable market shares (i.e., $\varepsilon_3 < \omega_1(\varepsilon_2)$), which are illustrated by Regions I and II in Figure 2a. In particular, $\Delta_m$ will not be formed in $\delta$-fashion in
symmetric markets (i.e., $\varepsilon_2 = \varepsilon_3 = 0$).  

Proposition 3: Under $A_0$, $\Delta^*$ and $W^*$ are given by:

$$
\Delta^* =
\begin{cases}
\{123\} & \text{if } \frac{5}{22} \leq \varepsilon_2 \leq \frac{1}{2} \\
\{1; 23\} \text{ or } \{13; 2\} & \text{if } \frac{7}{38} < \varepsilon_2 < \frac{5}{22} \text{ or if } \varepsilon_2 \leq \frac{7}{38} \text{ and } \varepsilon_3 \geq \omega_2 \\
\{1; 2; 3\} & \text{if } \varepsilon_2 \leq \frac{7}{38}; \varepsilon_3 < \omega_2;
\end{cases}
$$

(16)

$$
W^* =
\begin{cases}
\frac{3(a-c_1)^2}{8} & \text{if } \frac{5}{22} \leq \varepsilon_2 \leq \frac{1}{2} \\
\frac{(a-c_1)^2(8-8\varepsilon_2+11\varepsilon_2^2)}{18} & \text{if } \frac{7}{38} < \varepsilon_2 < \frac{5}{22} \text{ or if } \varepsilon_2 \leq \frac{7}{38} \text{ and } \varepsilon_3 \geq \omega_2 \\
\frac{(a-c_1)^2[15-10(\varepsilon_3+\varepsilon_2)-18\varepsilon_2\varepsilon_3+23(\varepsilon_2^2+\varepsilon_3^2)]}{32} & \text{if } \varepsilon_2 \leq \frac{7}{38}; \varepsilon_3 < \omega_2.
\end{cases}
$$

(17)

Propositions 2 and 3 lead directly to the following corollary:

Corollary 1: If $\varepsilon_2 \geq \frac{5}{22}$, $\Delta_m = \{123\}$ is both $\delta$-stable and socially optimal.

Hence, monopoly is both $\delta$-stable and optimal if cost savings are sufficiently large (i.e., $\varepsilon_2 \geq 5/22 > 1/6$). In such cases, no anti-trust regulation is needed as monopoly is the best market structure. These are illustrated in Figure 2b and in Example 1 below.

8 Rajan (1989) reported such symmetric case with $n = 3$ and $n = 4$.  

Figure 2. (a) The $\delta$-stability of $\Delta_m = \{123\}$; (b) the optimal partitions for $n = 3$. In both parts, the feasible region is in the area above the 45° line.
Example 1: For \((a; c; z) = (6; 0.5, 1, 1.2; 2, 2, 2)\), one has: \(\varepsilon_2 = 0.09\), \(\varepsilon_3 = 0.127\), \(\omega_1 = 0.19\), \(\omega_2 = 0.14\). By \(\varepsilon_3 < \omega_1\) and Proposition 2, \(\Delta_m\) is \(\delta\)-unstable. By \(\varepsilon_2 < 7/38\) and \(\varepsilon_3 < \omega_2\), and by (16), \(\Delta_0\) is optimal. Let costs be increased to \(c = (0.5, 1.9, 2)\) and \((a, z)\) be unchanged, then \(\varepsilon_2 = 0.26 > 5/22\). By Corollary 1, monopoly now is both \(\delta\)-stable and optimal.

Proposition 2 is proved by computing the MNBP defined in (5), which allows us to analyze the effects of merging costs (or costs of coalition formation) on the stability of each partition. For each \(S \subseteq N\), let \(MC_S\) denote its merging costs (synergy if negative). Since an analysis of \(MC_S > 0\) for \(S \neq N\) requires a separate study, we only study the effects of monopoly merging costs under the following assumption:

**A1 (Assumption 1):** \(MC_N \geq 0\), and \(MC_S = 0\) for all \(S \neq N\).

**Corollary 2:** Under \(A0\) and \(A1\), \(\Delta_m = \{123\}\) is \(\alpha\)-\((\delta\) stable \(\Leftrightarrow MC_N \leq [v(N) - MNBP_\alpha]\) \([v(N) - MNBP_\delta]\), where \(v(N)\), \(MNBP_\alpha\) and \(MNBP_\delta\) are given in \((A10-A16)\) in appendix.

Hence, large monopoly merging costs will weaken monopoly’s stability, and the difference between monopoly’s profits and its MNBP defines an upper bound for its merging costs above which they will destroy the formation of monopoly merger, see Zhao (2009) on the estimation of such merging costs.

Another advantage of the MNBP method is that it allows us to analyze whether a monopoly will remain stable in face of outside perturbations (i.e., its external stability).

**Corollary 3:** Under \(A0-A1\), an \(\alpha\)-\((\delta\) stable monopoly remains as a stable monopoly for small perturbations in market parameters \(\Leftrightarrow MC_N < [v(N) - MNBP_\alpha]\) \([v(N) - MNBP_\delta]\)).

In other words, a monopoly merger will unravel in the face of small perturbations if it

\footnote{Precisely, a stable \(\Delta_m\) in \((a, c, z) = t \in \mathbb{R}_+^7\), remains stable against small perturbations if there exists \(\varepsilon > 0\) such that \(\Delta_m\) is stable for all \(t' \in B_\varepsilon(t)\), where for \(t \in \mathbb{R}^7\), \(B_\varepsilon(t) = \{y \in \mathbb{R}^7 \mid ||t-y|| < \varepsilon\}\) and \(||t||^2 = \Sigma_i t_i^2\).}
is unstable or if it is stable with \(MC_N = [v(N) - MNB_P_2] (= [v(N) - MNB_P_3])\). It is straightforward to extend Corollaries 2 and 3 on merging costs and sensitivity to non-monopoly partitions, we therefore will skip such extensions in the rest of this paper.

We now study the stability of \(\Delta_1 = \{1, 23\}\), \(\Delta_2 = \{13, 2\}\), and \(\Delta_3 = \{12, 3\}\). Because the \(\alpha\), \(\gamma\) and \(\delta\)-stabilities for each \(\Delta_i (i = 1, 2, 3)\) are identical (though they are different for \(\Delta_m\) and for \(n > 3\)), there is no need to make such distinction here. The outsiders’ or the single firms’ post-merger profits at each \(\Delta_i\) are equal to

\[
\pi_i(\Delta_i) = \frac{(a - c_2)^2 (1 + \varepsilon_i)^2}{9}, \quad \pi_2(\Delta_2) = \frac{(a - c_2)^2 (1 - 2 \varepsilon_2)^2}{9}, \quad \pi_3(\Delta_3) = \frac{(a - c_2)^2 (1 - 2 \varepsilon_3)^2}{9}.
\]

(18)

Denote the merger’s gain for each \(S = \{12\}, \{13\}\) and \(\{23\}\) by

\[
d_{12} = \pi_{12}(\Delta_3) - (\pi_1 + \pi_2), \quad d_{13} = \pi_{13}(\Delta_2) - (\pi_1 + \pi_3), \quad d_{23} = \pi_{23}(\Delta_1) - (\pi_2 + \pi_3);
\]

(19)

and denote the efficient member’s share of the above gains by \(t \in [0, 1]\). Then, the three dimensional post-merger profit vector \(\lambda(t)\) for each \(\Delta_i\) can be given as

\[
\begin{align*}
\text{for } \Delta_1, & \quad \lambda_1 = \pi_{1}(\Delta_i), \quad \lambda_2 = \pi_2 + t \ d_{23}, \quad \text{and } \lambda_3 = \pi_3 + (1-t) \ d_{23};
\text{for } \Delta_2, & \quad \lambda_1 = \pi_1 + t \ d_{13}, \quad \lambda_2 = \pi_2(\Delta), \quad \text{and } \lambda_3 = \pi_3 + (1-t) \ d_{13}; \quad \text{and} \quad \lambda_1 = \pi_1 + t \ d_{12}, \quad \lambda_2 = \pi_2 + (1-t) \ d_{12}, \quad \lambda_3 = \pi_3(\Delta_3).
\end{align*}
\]

(20)

The stability of each \(\Delta_i\) requires two preconditions: its merger \(S\) is profitable (i.e., \(d_S > 0\)) and monopoly is unprofitable due to high merging costs (i.e., \((v(N) - MC_N) < \Sigma_j\)). Under these two preconditions, the stability of each \(\Delta_i\) is determined by the magnitude of cost differential \(\varepsilon_i\) and by the size of the share \(t\), which are captured by a critical level \(\mu_i(\varepsilon_i, t)\) given in appendix: \(\mu_1(\varepsilon_2, t)\) in (C14), \(\mu_2(\varepsilon_2, t)\) in (C16), and \(\mu_3(\varepsilon_3, t) = \mu_2(\varepsilon_3, t)\).

**Proposition 4:** Given \((a, c, z) \in R^7_{++}\), suppose \(\Sigma_j > (v(N) - MC_N)\) and \(d_S > 0\) for each \(S = \{12\}, \{13\}\) and \(\{23\}\).
12, 13, 23. Under $A0$ and $A1$, the following three claims hold:

(i) $\Delta_1 = \{1; 23\}$ with $\lambda(t)$ is stable $\iff \varepsilon_3 \leq \mu_1(\varepsilon_2, t)$; $\varepsilon_3 \leq \mu_1(\varepsilon_2, t)$ holds if $0 \leq \varepsilon_2 \leq 1/11$; and $\varepsilon_3 > \mu_1(\varepsilon_2, t)$ holds if $113/316 < \varepsilon_2 \leq \frac{1}{2}$.

(ii) $\Delta_2 = \{13; 2\}$ with $\lambda(t)$ is stable $\iff \varepsilon_3 \leq \mu_2(\varepsilon_2, t)$. Let $e_2(t) = (2t-9)/[14(2t-3)]$.

Then, $\varepsilon_3 \leq \mu_2(\varepsilon_2, t)$ holds if $0 \leq \varepsilon_2 < 1/11$; $\varepsilon_3 > \mu_2(\varepsilon_2, t)$ holds if $e_2(t) < \varepsilon_2 \leq \frac{1}{2}$.

(iii) $\Delta_3 = \{12; 3\}$ with $\lambda(t)$ is stable $\iff \varepsilon_2 \leq \mu_3(\varepsilon_3, t)$; and it holds if $0 \leq \varepsilon_3 \leq 3/14$.

To see the intuition of these results, let us focus our attention on $\Delta_1 = \{1; 23\}$ with $\lambda(t) = \{\pi_1(\Delta_1), \pi_2 + td_{23}, \pi_3 + (1-t)d_{23}\}$. Because $\Delta_0$ and $\Delta_m$ are ruled out by preconditions and $\Delta_2 = \{13, 2\}$ has the same profits of $\Delta_1$, the only possible deviation is $\Delta_3 = \{12, 3\}$, when firm 2 breaks up $T = \{23\}$ and merges with firm 1. In this light, part (i) is transparent: since a larger share of $d_{23}$ by firm 2 or a smaller $\varepsilon_3$ makes the deviation from $\Delta_1$ to $\Delta_3$ less profitable for $S = \{12\}$, $\Delta_1$ with $\lambda(t)$ will be stable with a smaller $\varepsilon_3$ or a larger $t$, or precisely $\varepsilon_3 \leq \mu_1(\varepsilon_2, t)$ holds (note $\mu_1(\varepsilon_2, t)$ is increasing in $t^{10}$). Indeed, inverting $t$ in $\mu_1(\varepsilon_2, t) = \varepsilon_3$ yields

$$t_1(\varepsilon_2, \varepsilon_3) = \frac{1}{2} \frac{7\varepsilon_3^2 - 9\varepsilon_2^2 + 54\varepsilon_2\varepsilon_3 + 14\varepsilon_3 + 22\varepsilon_2 - 9}{45\varepsilon_3^2 + 13\varepsilon_2^2 - 54\varepsilon_2\varepsilon_3 - 18\varepsilon_3 + 14\varepsilon_2 + 1},$$

which leads to an alternative characterization for the stability of $\Delta_1$ given below:

**Corollary 4:** Let $\varepsilon_2 \in [1/11, 113/316]$. Then, $\Delta_1$ with $\lambda(t)$ is stable $\iff t \geq t_1(\varepsilon_2, \varepsilon_3)$.

Hence, internal cooperation within a merger (or the share $t$) is a key determinant for merger stability. The set of markets with a stable $\Delta_1$ is given by Region I in Figure 3a, where the feasible region is bounded by $\theta_0$ and $\theta_0$ (see (A6) in appendix) due to preconditions. Similarly, Region I in Figures 3b and 4 represent markets in which $\Delta_2$ and $\Delta_3$ are stable.

---

10 $\mu_1(\varepsilon_2, t)$ could be increasing or decreasing in $\varepsilon_2$ (see Figure 3a), so the effects of $\varepsilon_2$ are ambiguous.
respectively. The stability of $\Delta_1$ is also illustrated numerically by Example 2 below:

**Example 2:** Let $(a; c; z) = (6; 0.5; 1.05; 2.46; 3, 3, 3)$, then $\varepsilon_2 = 0.1$, $\varepsilon_3 = 0.356$, $(\pi_1, \pi_2, \pi_3) = (4.01, 2.11, 0.002)$, $(\overline{\pi}_1, \overline{\pi}_2) = (4.067, 2.152)$, $t_1(\varepsilon_2, \varepsilon_3) = 0.12$. Hence, $t = 0.1$ or $\lambda(0.1) = (4.067, 2.114, 0.038)$ is unstable, and $t = 0.2$ or $\lambda(0.2) = (4.067, 2.118, 0.034)$ is stable. If $c_3$ rises so $\varepsilon_3 = 0.358$, then $t_1(\varepsilon_2, \varepsilon_3)$ becomes $t_1(0.1, 0.358) = 0.45$, so $t = 0.2$ or $\lambda(0.2)$ is now unstable.

**Figure 3.** (a) Stabil ity of $\Delta_1$, feasible region is $\text{Max}\{\varepsilon_2, \theta_6\} \leq \varepsilon_3 \leq \theta_0$; (b) stability of $\Delta_2$, feasible region is $\text{Max}\{\varepsilon_2, \theta_4\} \leq \varepsilon_3 \leq \theta_0$. In both cases, $t$ is set at 0.

**Figure 4.** The Stability of $\Delta_3$, feasible region is $\varepsilon_2 \leq \varepsilon_3 \leq \text{Min}\{\theta_0, \theta_2\}$; $t$ is set at 0, and $\mu_3(\varepsilon_3, t)$ is represented by $\mu_5(\varepsilon_2, 0)$ and $\mu_{50}(\varepsilon_2, 0)$.

Finally, we now know that the original Cournot structure $\Delta_0 = \{1; 2; 3\}$ is stable if

---

11 Figure 4 shows the two solutions of $\varepsilon_2$ in $\varepsilon_2 = \mu_3(\varepsilon_3, t)$: $\varepsilon_2 = \mu_5(\varepsilon_2, t) \geq \varepsilon_3 = \mu_{50}(\varepsilon_2, t)$, and the minimum of $\mu_3(\varepsilon_2, t)$ is $\varepsilon_2^\prime(t) = \text{Min}\{\mu_3(\varepsilon_2, t) | \varepsilon_3\} = \mu_3(\varepsilon_3^\prime(t), t)$, where $\varepsilon_3^\prime(t) = 0.179$ and $\varepsilon_2^\prime(t) = 0.293$. Then, part (iii) becomes: $\lambda(t)$ is stable if (a) $\varepsilon_2 \leq \varepsilon_2^\prime(t)$, or (b) $\varepsilon_2 > \varepsilon_2^\prime(t), \varepsilon_3 \leq \varepsilon_2^\prime(t), \varepsilon_3 \leq \mu_{50}(\varepsilon_2, t)$; or (c) $\varepsilon_2 > \varepsilon_2^\prime(t), \varepsilon_3 > \varepsilon_2^\prime(t), \varepsilon_3 \geq \mu_3(\varepsilon_2, t)$. 

---

15
and only if none of the four mergers are profitable, which is given by the corollary below:

**Corollary 5:** \( \Delta_0 = \{1; 2; 3\} \) is stable if and only if the following (22) holds:

\[
\theta_2 < \varepsilon_3 < \theta_4, \text{ and } (v(N) - MC_N) < (\pi_1 + \pi_2 + \pi_3).
\]  

As shown in Figure 5, \( \Delta_0 \) will always be unstable if \( \varepsilon_2 \geq 1/14 \), because \( \theta_4 < \theta_2 \) (or \( d_{13} > 0 \)) holds for all \( \varepsilon_2 \in [1/14, 1/2] \).

To summarize our conclusions for three-firm Cournot oligopolies, monopoly will be the solution if it is profitable and the two inefficient firms are sufficiently small (i.e., \( \varepsilon_2 \) is large).\(^\text{(12)}\) When monopoly is ruled out by high merging costs, a profitable duopoly will be the solution if cost differentials are small (i.e., \( \varepsilon_3 \) is small). Finally, Cournot equilibrium will be the solution if none of the four mergers is profitable.

5. Extensions to markets with \( n \) firms

Recall that monopoly’s \( \alpha \)- and \( \gamma \)-stabilities are identical, so we only need to check the

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\(^\text{(12)}\) Otherwise, there will be no solution, as market structure could move cyclically among the five partitions (i.e., \( \Delta_\mu \) will be formed, broken up, and then formed again). This generates a new form of business
\(\alpha\) and \(\delta\)-stabilities of \(\Delta_m\). Let the MNBP against \(\alpha\)- and \(\delta\)-deviations from \(\Delta_m\) be given by

\[
\text{MNBP}_\alpha = \{\min \sum x_i \mid x \in R^n_+; \sum_{i \in S} x_i \geq \phi(S, B_\alpha(S)), \text{ all } S \neq N\},
\]

(23)

\[
\text{MNBP}_\delta = \{\min \sum x_i \mid x \in R^n_+; \sum_{i \in S} x_i \geq \phi(S, B_\delta(S, \Delta_m)), \text{ all } S \neq N\},
\]

(24)

where \(\phi(S, \Delta), B_\alpha(S) = \{S, (i_1), \ldots, (i_{n-k})\}\) and \(B_\delta(S, \Delta_m) = \{S, N/S\}\) are given in (7)-(11). For a non-monopoly partition \(\Delta \neq \Delta_m\), let

\[
Y_\alpha(\Delta) = \{x \in R^n_+ \mid \sum_{i \in S} x_i \geq \phi(S, B_\alpha(\Delta, S)) \text{ for all } S \neq N, S \notin \Delta\},
\]

\[
Y_\gamma(\Delta) = \{x \in R^n_+ \mid \sum_{i \in S} x_i \geq \phi(S, B_\gamma(\Delta, S)) \text{ for all } S \neq N, S \notin \Delta\},
\]

\[
Y_\delta(\Delta) = \{x \in R^n_+ \mid \sum_{i \in S} x_i \geq \phi(S, B_\delta(\Delta, S)) \text{ for all } S \neq N, S \notin \Delta\}.
\]

(25)

denote the sets of profit vectors immune to the \(\alpha\), \(\gamma\), and \(\delta\)-deviations, respectively, and let

\[
\text{MNBP}_\alpha(\Delta) = \{\min x_i \mid x \in Y_\alpha(\Delta)\},
\]

\[
\text{MNBP}_\gamma(\Delta) = \{\min x_i \mid x \in Y_\gamma(\Delta)\}, \text{ and,}
\]

\[
\text{MNBP}_\delta(\Delta) = \{\min x_i \mid x \in Y_\delta(\Delta)\}
\]

(26)

denote the corresponding values of MNBP.

Proposition 5 below provides a computational characterization for each stable partition in a general linear oligopoly.

**Proposition 5:**

i) Under \(A0\) and \(A1\), the monopoly merger is \(\alpha\)- (\(\delta\)-) stable if and only if \((v(N) - MC_N) \geq \text{MNBP}_\alpha (\text{MNBP}_\delta)\).

ii) Given \(\Delta \neq \Delta_m\), assume \(A0\), \(A1\), and \(\Sigma_{S \in A} \tilde{\alpha}_S(\Delta) > (v(N) - MC_N)\). Then, \(\Delta\) is \(\alpha\)- (\(\gamma\); \(\delta\)-) stable if and only if \(\Sigma_{S \in A} \tilde{\pi}_S(\Delta) \geq \text{MNBP}_\alpha(\Delta) (\text{MNBP}_\gamma(\Delta); \text{MNBP}_\delta(\Delta))\).

By part i), monopoly will be formed in the \(\alpha\)- (\(\delta\)-) fashion if its profits after paying the merging costs exceed \(\text{MNBP}_\alpha (\text{MNBP}_\delta)\). By part ii), mergers in \(\Delta \neq \Delta_m\) will be formed in cycle if partition changes take a fixed length of time.
the $\alpha$- ($\gamma$; $\delta$) fashion if the current total profits are no less than $\text{MNBP}_\alpha$ ($\text{MNBP}_\gamma$; $\text{MNBP}_\delta$).

Such computational results also hold in n-firm non-linear oligopolies, as one could compute the values of MNBP in (23)-(24) and (26) and then compare them with $v(N)$ or $\Sigma_{S \in \Delta} \pi_S(\Delta)$.

We now characterize a stable monopoly in a class of asymmetric linear oligopolies, which is similar to the dominant cartel model with a finite number of Cournot fringe firms. We leave the stability for $\Delta \neq \Delta_m$ in these oligopolies to future studies, whose computations (those given in part ii) of Proposition 5) are beyond the interests of this paper.

Precisely, the model is given by $(a, c, z) \in \mathbb{R}^{2n+1}_{++}$ with $c_j \equiv c_2 \geq c_1$ for all $j \geq 3$. Let

$$\varepsilon = \varepsilon_2 = (c_2-c_1)/(a-c_1) \quad (27)$$

be firm 1’s relative cost savings over the fringe firms ($j \geq 2$, all $j$), and define $\omega_0$ by

$$\omega_0 = \omega_0(n) = (4n-9)/(8n-6). \quad (28)$$

**Proposition 6:** Given $(a, c, z) \in \mathbb{R}^{2n+1}_{++}$ with $c_j \equiv c_2 \geq c_1$, all $j \geq 3$. Under $A0$, the following three claims hold: (i) monopoly is $\alpha$-stable for all $n$; (ii) monopoly is $\delta$-stable for $n = 2$; and (iii) monopoly is $\delta$-stable for $n \geq 3$ if and only if $\varepsilon \geq \omega_0$ holds.

By part (iii) and by the fact that $\omega_0(n)$ is increasing in $n$, larger cost savings will strengthen the monopoly’s $\delta$-stability, and more firms will harm its $\delta$-stability. Since monopoly is optimal if cost savings are sufficiently large, the stability and optimality of the monopoly merger go hand in hand in markets with large cost savings.

In the symmetric case (i.e., $\varepsilon=0$), monopoly will be formed in the $\alpha$-fashion and not be formed in the $\delta$-fashion (except with $n=2$), which can be given as a corollary below.

**Corollary 6:** Given $(a, c, z) \in \mathbb{R}^{2n+1}_{++}$ with $c_j \equiv c_1$, all $j$. Under $A0$, the following two
claims hold: (i) $X_\alpha(\Delta m) \neq \emptyset$ for all $n$; and ii) $X_\delta(\Delta m) \neq \emptyset$ for $n = 2$, and $= \emptyset$ for all $n \geq 3$.

It is interesting to note that the above symmetric results have been reported in the study of a common pool resource problem by Funaki and Yamato (1999). With a production function $f(Q) = (a-Q)Q$, their Theorem 4 becomes $X_\alpha(\Delta m) \neq \emptyset$, and their Theorem 6 becomes $X_\delta(\Delta m) = \emptyset$ for all $n \geq 4$. It remains to be seen if $X_\delta(\Delta m) = \emptyset$ also holds for $n = 3$ in their model.

It will be useful and workable to extend the above results to more general oligopolies in future studies. One extension is a linear oligopoly with two types of firms (i.e., $m$ firms with $c_1$, and $n$ firms with $c_2$), and another is an oligopoly with non-linear demand and linear costs. These extensions will generate enough results worthy of two or more articles.

6. Conclusion and discussion

We have defined and provided a computational characterization for a stable market structure, which is a set of simultaneous mergers that are free of subsequent breakups or new mergers. Our computational results also hold in general oligopolies and normal formal games with transferable utilities, which can be obtained by computing the MNBP for each given partition and then comparing the value against its current total payoffs.

John Nash suggested in his 1998 Cowles talk that “there isn't any theory yet that seems acceptable as providing a solution concept” in cooperative games with at least three players. The author believes that a stable partition or a set of stable merger contracts provides an answer to Nash’s challenging comment, because it rules out all possible deviations.

Applying the technique to three-firm linear Cournot oligopolies, we have obtained a complete answer to the problem, namely, monopoly will be the solution if its merging cost is
sufficiently low and cost differentials are sufficiently large. When monopoly is ruled out by high merging costs, a profitable two-member merger will be the solution if its efficient member’s share of the merger’s gain are sufficiently large and cost differentials are sufficiently small, and Cournot equilibrium will be the solution if none of the four mergers are profitable.

Readers are encouraged to apply our technique to more general oligopolies or normal form games to obtain new understandings about stable partitions, and to empirically estimate the size of merging costs which is a crucial determinant for the stability of market structures.

APPENDIX

We first compute the involved MNBP. Lemma 1 provides the MNBP used in Proposition 6, and Lemmas 2-3 provides the MNBP and profitability in Propositions 2-4.

Given \((a,c,z)\in\mathbb{R}^{m+1}_{++}\) with \(c_j = c_2 \geq c_1\) for \(j \geq 3\), let \(\varepsilon\) be given by (27), and define \(n_0\) by

\[
n_0 = \frac{9}{1+4\varepsilon} - 4; \quad n_0 \in [-1, 5].
\]  

(A1)

**Lemma 1**: Assume \(c_j = c_2 \geq c_1\) for \(j \geq 3\). Then, (23) and (24) are given by:

\[
\begin{align*}
\text{MNBP}_\alpha &= \begin{cases} 
(a-c_1)^2 \frac{\{(1+(n-1)\varepsilon)^2 + (n-1)(1-2\varepsilon)^2\}/(n+1)^2}{(n+1)^2} & \text{if } \varepsilon \leq 1/8 \text{ and } n \leq n_0 \\
(a-c_1)^2 \frac{(1+\varepsilon)^2/9 + (1-2\varepsilon)^2/(n+1)^2}{(n+1)^2} & \text{if } 1/8 < \varepsilon \text{ and } n \leq n_0
\end{cases} \quad \text{(A2)}
\end{align*}
\]

\[
\begin{align*}
\text{MNBP}_\delta &= (a-c_1)^2 \frac{(1+\varepsilon)^2 + (1-2\varepsilon)^2}{9(n-1)} \quad \text{if } 6 \leq n; \quad \text{(A3)}
\end{align*}
\]

In symmetric case with \(c_j = c_1\), all \(j\) (or \(\varepsilon=0\)), the above (A2)-(A3) become:

\[
\begin{align*}
\text{MNBP}_\alpha &= \text{MNBP}_{\alpha}(a, c, z) = \begin{cases} 
\frac{n(a-c_1)^2}{(n+1)^2} & \text{if } 2 \leq n \leq 5 \\
\frac{n(a-c_1)^2}{[9(n-1)]} & \text{if } 6 \leq n; \\
\end{cases} \quad \text{(A4)}
\end{align*}
\]

\[
\begin{align*}
\text{MNBP}_\delta &= n(a-c_1)^2 / 9. \quad \text{(A5)}
\end{align*}
\]

Let \(\Delta_0 = \{1, 2, 3\}\), \(\Delta_1 = \{1, 23\}\), \(\Delta_2 = \{13, 2\}\), \(\Delta_3 = \{12, 3\}\), \(\Delta_m = \{(1, 2, 3)\}\), and \(\pi_i = \)
\( \pi(\hat{\Theta}) = v_1 \) and \( \tilde{\pi}(\Delta) = \{ \tilde{\pi}_S(\Delta) \mid S \in \Delta \} \) be pre- and post-merger profits. For \((a, c_1, c_2, c_3; z_1, z_2, z_3) \in \mathbb{R}_+^7\), let \( \varepsilon_2 \) and \( \varepsilon_3 \) be given by (13), \( \theta_i \) be defined as

\[
\theta_0 = \frac{1 + \varepsilon_2}{3}, \quad \theta_1 = 15\varepsilon_2 - 1, \quad \theta_4 = \frac{1 + \varepsilon_2}{15}, \quad \theta_6 = \frac{1 + 13\varepsilon_2}{15}, \quad \text{and}
\]

\[
d_{12} = \tilde{\pi}_{12}(\Delta_3) - (\pi_1 + \pi_2), \quad d_{13} = \tilde{\pi}_{13}(\Delta_2) - (\pi_1 + \pi_3), \quad d_{23} = \tilde{\pi}_{23}(\Delta_1) - (\pi_2 + \pi_3)
\]

be the \( S' \) increase of joint profits for \( S = 12, 13, \) and \( 23. \)

**Lemma 2:**

(I) \( d_{12} > 0 \Leftrightarrow \varepsilon_3 < \theta_2; \) (II) \( d_{13} > 0 \Leftrightarrow \varepsilon_3 > \theta_4; \) and (III) \( d_{23} > 0 \Leftrightarrow \varepsilon_3 > \theta_6. \)

As shown in Figure 5, a merger is profitable \( \Leftrightarrow \) its cost savings are sufficiently large (note a larger \( \varepsilon_2 \) or \( \varepsilon_3 \) represents larger cost saving). Let \( \rho_i, v_i \) and \( y_i \) be given by

\[
\rho_0 = -1 + \sqrt{2 - 2\varepsilon_2 + 5\varepsilon_2^2}, \quad \rho_1 = (5 - 11\varepsilon_2)/11, \quad \rho_2 = -1 + 27\varepsilon_2 - 4\sqrt{3\varepsilon_2 + 42\varepsilon_2^2},
\]

\[
\rho_3 = \frac{19 + 27\varepsilon_2 + 4\sqrt{17 - 125\varepsilon_2 + 218\varepsilon_2^2}}{89}, \quad \rho_4 = \frac{19 + 27\varepsilon_2 - 4\sqrt{17 - 125\varepsilon_2 + 218\varepsilon_2^2}}{89}, \quad \rho_5 = \frac{125 - 3\sqrt{89}}{436} \approx 0.22;
\]

\[
v_1 = \pi_1 = (a - c_1)^2(1 + \varepsilon_2 + \varepsilon_3)^2/16, \quad v_2 = \pi_2 = (a - c_1)^2(1 - 3\varepsilon_2 + \varepsilon_3)^2/16, \quad v_3 = \pi_3 = (a - c_1)^2(1 - 3\varepsilon_3 + \varepsilon_2)^2/16;
\]

\[
v_{12} = (a - c_1)^2(1 + \varepsilon_3)^2/9, \quad v_{13} = v_1^0 = \tilde{\pi}_1(\Delta_1) = (a - c_1)^2(1 + \varepsilon_2)^2/9, \quad v_{23} = v_2^0 = \tilde{\pi}_2(\Delta_2) = (a - c_1)^2(1 - 2\varepsilon_2)^2/9, \quad \text{and}
\]

\[
v_{123} = v(NG) = (a - c_1)^2/4;
\]

\[
y_1 = (v_{12} + v_{13} - v_{23})/2, \quad y_2 = (v_{12} + v_{23} - v_{13})/2, \quad y_3 = (v_{12} + v_{23} - v_{13})/2.
\]

**Lemma 3:** Under parts (ii) and (iii) of A0, (23) and (24) are given by:

\[
MNB_{\delta} = \begin{cases} 
    v_{13} + v_{23} + v_3^0 & \text{if } \varepsilon_3 \leq \rho_0 \\
    v_{12} + v_3^0 & \text{if } \varepsilon_3 > \rho_0.
\end{cases}
\]

For \( \varepsilon_2 \leq \frac{1}{14} \), \( MNB_{\alpha} = \begin{cases} 
    v_1 + v_2 + v_3 & \text{if } \varepsilon_3 < \theta_4 \\
    v_2 + v_{13} & \text{if } \varepsilon_3 \geq \theta_4.
\end{cases}
\]
For $\frac{1}{14} \leq \varepsilon_2 \leq \frac{1}{17}$, \( \text{MNBP}_\alpha = v_2 + v_{13} \); (A14)

For $\varepsilon_2 \geq \frac{1}{17}$, $\varepsilon_3 \leq \theta_0$, \( \text{MNBP}_\alpha = \begin{cases} 
\varepsilon_2 + v_{13} & \text{if } \frac{1}{17} \leq \varepsilon_2 \leq \frac{16}{77} \\
\varepsilon_2 + v_{13} & \text{if } \varepsilon_3 \leq \rho_1; \frac{16}{77} \leq \varepsilon_2 \leq \frac{5}{22} \\
v_3 + v_{12} & \text{if } \varepsilon_3 > \rho_1; \frac{5}{22} \leq \varepsilon_2 \leq \frac{11}{17} \\
v_3 + v_{12} & \text{if } \varepsilon_2 \geq \frac{5}{22}, 
\end{cases}\) (A15)

For $\varepsilon_2 \geq \frac{1}{17}$, $\varepsilon_3 > \theta_0$, \( \text{MNBP}_\alpha = \begin{cases} 
\varepsilon_3 + v_{12} & \text{if } \varepsilon_3 \leq \rho_3 \text{ for } \frac{16}{77} \leq \varepsilon_2 \leq \frac{5}{22} \\
v_3 + v_{12} & \text{if } \varepsilon_3 > \rho_3 \text{ or } \varepsilon_3 \geq \rho_5 \\
v_3 + v_{12} & \text{for } \varepsilon_2 > \rho_5 \approx 0.22. 
\end{cases}\) (A16)

Our results are proved in the following order with increasing complexity: (a) Proposition 5; (b) Lemma 1 and Proposition 6; (c) Lemmas 2-3; and (d) Propositions 2-4.

Proof of Proposition 5: Part i). Suppose $(v(N)-MC_N) \geq \text{MNBP}_\alpha$. Let $x$ solve (23), and $\lambda$ be given by $\lambda_i = x_i + (v(N)-MC_N - \text{MNBP}_\alpha)/n$, all $i$. By (23), $\sum_{j \in S} \lambda_j \geq \sum_{j \in S} x_j \geq \phi(B_{\gamma}(\Delta_m,S), S)$ for all $S \neq N$. Hence, $\Delta_m$ with $\lambda$ is $\alpha$-stale when $(v(N)-MC_N) \geq \text{MNBP}_\alpha$. If $(v(N)-MC_N) < \text{MNBP}_\alpha$, no allocation of $(v(N)-MC_N)$ could satisfy all coalitions, therefore the monopoly merger with any $\lambda$ will not be $\alpha$-stale. The proof for part ii) is similar. Q.E.D

Before we prove Lemma 1, we first compute the simple case of (A4)-(A5).

Derivation of (A4)-(A5): The proof completes in six steps.

**Step 1.** Determine $\phi(\beta_\alpha(S), S)$ and $\phi(\beta_\delta(\Delta_m,S), S)$. For each $1 \leq k \leq (n-1)$, let $S(k) = \{T \subseteq N \mid |T| = k\}$. By A0 and by symmetry, its $\alpha$- and $\delta$-profits are:

$$
\phi(\beta_\alpha(S), S) = \phi(\beta_\alpha(\Delta_m,S), S) = v_\alpha(k) = (a-c_1)^2/(n-k+2)^2, 
$$
(B1)

$$
\phi(\beta_\delta(\Delta_m,S), S) = v_\delta(k) = (a-c_1)^2/9.
$$
(B2)

**Step 2.** Determine $MV_\alpha(k)$ and $MV_\delta(k)$. For each $k = 1, \ldots, n-1$, define
\( MV_\alpha(k) = \{ \min \sum_{i \in N} x_i \mid x \in \mathbb{R}^n; \sum_{s} x_i \geq v_\alpha(k), \text{ for all } S \in S(k) \}, \) and
\( MV_\delta(k) = \{ \min \sum_{i \in N} x_i \mid x \in \mathbb{R}^n; \sum_{s} x_i \geq v_\delta(k), \text{ for all } S \in S(k) \}. \) (B3)

There are \( \binom{n}{k} \) constraints, and each \( x_i \) appears \( \binom{n-1}{k-1} \) times. Summing up, we have
\( \binom{n-1}{k-1} \sum x_i \geq \binom{n}{k} v_\alpha(k), \) and \( \binom{n-1}{k-1} \sum x_i \geq \binom{n}{k} v_\delta(k), \) or \( \sum x_i \geq n v_\alpha(k)/k, \) and \( \sum x_i \geq n v_\delta(k)/k. \)

Hence, define
\( MV_\alpha(k) = n(a-c_1)^2/[k(n-k+2)^2], \) and \( MV_\delta(k) = n(a-c_1)^2/[9k], \) (B4)
and the minimum solutions are \( x_i = (a-c_1)^2/[k(n-k+2)^2], \) all \( i; \) and \( y_i = (a-c_1)^2/(9k), \) all \( i. \)

**Step 3.** Determine the maximum of \( MV_\alpha(k) \) and \( MV_\delta(k). \) By \( MV_\delta(k)' < 0, \)
\[ \text{Max } \{ MV_\delta(k) \mid 1 \leq k \leq (n-1) \} = MV_\delta(1) = n(a-c_1)^2/9. \] (B5)

By \( MV_\alpha(k)' \leq 0 \) if \( k \leq (n+2)/3, \) and \( > 0 \) if \( k > (n+2)/3, \) we have
\[ \text{Max } \{ MV_\alpha(k) \mid 1 \leq k \leq (n-1) \} = \text{Max } \{ MV_\alpha(1), MV_\alpha(n-1) \}. \]

Define \( f(n) = MV_\alpha(1)-MV_\alpha(n-1), \) which has the same sign as \( g(n) = 9(n-1) - (n+1)^2. \) Since \( g(n) \) is \( \cap \)-shaped and has two roots: \( n = 2 < 5, \) one has:
\[ \text{Max } \{ MV_\alpha(k) \mid 1 \leq k \leq (n-1) \} = \begin{cases} 
\frac{n(a-c_1)^2}{(n+1)^2} & \text{if } 2 \leq n \leq 5 \\
\frac{n(a-c_1)^2}{9(n-1)} & \text{if } 6 \leq n.
\end{cases} \] (B6)

**Step 4.** Enlarge the feasible sets for \( MV_\alpha(k) \) and \( MV_\delta(k). \) For \( k = 1, \ldots, n-1, \) let
\( FR_\alpha(k) = \{ x \in \mathbb{R}^n \mid \sum_{s} x_i \geq v_\alpha(k), \text{ for all } S \in S(k) \}, \) and
\( FR_\delta(k) = \{ x \in \mathbb{R}^n \mid \sum_{s} x_i \geq v_\delta(k), \text{ for all } S \in S(k) \} \) (B7)
denote the feasible sets in (B3), and define
\( FR_\alpha(k)^* = \{ x \in \mathbb{R}^n \mid \sum x_i \geq MV_\alpha(k) \}, \) and \( FR_\delta(k)^* = \{ x \in \mathbb{R}^n \mid \sum x_i \geq MV_\delta(k) \}. \) (B8)

By the arguments in Step 2, the following relations hold for \( k = 1, \ldots, n-1. \)
\[ FR_\alpha(k) \subseteq FR_\alpha(k)^* \text{ and } FR_\delta(k) \subseteq FR_\delta(k)^*. \] (B9)

**Step 5.** Enlarge the feasible regions for MNBP_\alpha and MNBP_\delta. The feasible regions for MNBP_\alpha and MNBP_\delta as given in (23) and (24) are:
By the maximality in Step 3 and by (B8), one has
\[
FR_\alpha^* = \bigcap_{i=1}^{n-1} FR_\alpha(k)^* = \{x \in \mathbb{R}^n \mid \sum_{i \in S} x_i \geq \text{Max} \{MV_\alpha(1), MV_\alpha(n-1)\} \}
\]
\[
= \begin{cases} 
\{x \in \mathbb{R}^n \mid \sum_{i \in S} x_i \geq MV_\alpha(1)\} & \text{if } 2 \leq n \leq 5 \\
\{x \in \mathbb{R}^n \mid \sum_{i \in S} x_i \geq MV_\alpha(n-1)\} & \text{if } 6 \leq n;
\end{cases}
\]
(B12)
\[
FR_\delta^* = \bigcap_{i=1}^{n-1} FR_\delta(k)^* = \{x \in \mathbb{R}^n \mid \sum_{i \in S} x_i \geq MV_\delta(1)\}.
\]
(B13)

It follows from (B9)-(B13), the following relations hold:

\[
FR_\alpha \subseteq FR_\alpha^* = \bigcap_{i=1}^{n-1} FR_\alpha(k)^*, \text{ and } FR_\delta \subseteq FR_\delta^* = \bigcap_{i=1}^{n-1} FR_\delta(k)^*.
\]
(B14)

**Step 6.** Determine MNBP_\alpha and MNBP_\delta. Observe that the minimum value of
\[
\{\text{Min}_i x_i \mid x \in FR_\delta^*\}
\]
is equal to
\[
\{\text{Min}_i x_i \mid x \in FR_\delta^*\} = MV_\delta(1) = n(a-c_1)^2/9.
\]
(B15)

By the maximality in Step 3, the symmetric minimum solution (i.e., \(x_i = (a-c_1)^2/9\), all \(i\)) is included in all \(FR_\delta(k)\) and therefore also in \(FR_\delta\). Since a global min must be a local min when it is locally feasible, it follows from (B14) and (B15) that

\[
\text{MNBP}_\delta = \{\text{Min}_i x_i \mid x \in FR_\delta\} = \{\text{Min}_i x_i \mid x \in FR_\delta^*\} = n(a-c_1)^2/9.
\]

Using similar arguments, one obtains:  \(\text{MNBP}_\alpha = \{\text{Min}_i x_i \mid x \in FR_\alpha\} = \{\text{Min}_i x_i \mid x \in FR_\alpha^*\}\)

\[
= \begin{cases} 
(n(a-c_1)^2 / (n+1)^2) & \text{if } 2 \leq n \leq 5 \\
(n(a-c_1)^2 / [9(n-1)]) & \text{if } 6 \leq n;
\end{cases}
\]
which leads to (A5).

Q.E.D

**Proof of Lemma 1:** This proof is more involved, because there are now two types of mergers: type I mergers (1 \(\not\in S\)) and type II mergers (1 \(\in S\)). Let \(\varepsilon = (c_2-c_1)/(a-c_1)\). We first prove (A3), which involves exactly the same above six steps. The relevant formulae are:
\[
\phi(\beta_\delta(S), S) = v_\delta(k) = v^I_\delta(k) = \frac{(a-c_1)^2(1-2\varepsilon)^2}{9} \quad \text{for type I mergers, and}
\]
\[
v_\delta(k) = v^II_\delta(k) = \frac{(a-c_1)^2(1+\varepsilon)^2}{9} \quad \text{for type II mergers, } k = 1, \ldots, n-1;
\]

\[
\text{MV}_\delta(k) = \{\min \sum_{i \in \mathcal{N}} x_i \mid x \in \mathbb{R}^n_+ \; ; \sum_{i \in S} x_i \geq v_\delta(k), \text{ for all } S \in S(k)\}. \quad (B17)
\]

Summing up the constraints, one has \(\sum_{k=1}^{n-1} \sum_{i \in \mathcal{N}} x_i \geq \left(\sum_{k=1}^{n-1} v^I_\delta(k) + \left(\sum_{k=1}^{n-1} v^II_\delta(k)\right)\right)\), leading to

\[
\text{MV}_\delta(k) = \frac{(a-c_1)^2}{9} \left[\frac{(1+\varepsilon)^2 + (n-k)(1-2\varepsilon)^2}{k}\right], \quad (B18)
\]
whose minimum solution is given by

\[
x_i = \frac{(a-c_1)^2}{9} \left[\frac{(1+\varepsilon)^2 - (k-1)(1-2\varepsilon)^2}{k}\right], \quad \text{all } i \geq 2. \quad (B19)
\]

\[
\text{Max } \{\text{MV}_\delta(k) \mid 1 \leq k \leq (n-1)\} = \text{MV}_\delta(1) = \frac{(a-c_1)^2}{9} \left[\frac{(1+\varepsilon)^2 + (n-1)(1-2\varepsilon)^2}{(n-k+2)^2}\right]. \quad (B20)
\]

\[
\text{FR}_{\delta} = \cap_{i=1}^{n-1} \text{FR}_{\delta}(k) \subseteq \text{FR}^*_\delta = \cap_{i=1}^{n-1} \text{FR}^*_\delta(k) = \{x \in \mathbb{R}^n_+ \mid \sum_{i \in \mathcal{N}} x_i \geq \text{MV}_\delta(1)\}. \quad (B21)
\]

\[
\{\min \sum_{i \in \mathcal{N}} x_i \mid x \in \text{FR}^*_\delta\} = \text{MV}_\delta(1) = \frac{(a-c_1)^2}{9} \left[\frac{(1+\varepsilon)^2 + (n-1)(1-2\varepsilon)^2}{(n-k+2)^2}\right]. \quad (B22)
\]

The solution given by (B19) for \(k = 1\) is included in \(\text{FR}_{\delta}\). Therefore,

\[
\text{MNBP}_{\delta} = \{\min \sum x_i \mid x \in \text{FR}_{\delta}\} = \{\min \sum x_i \mid x \in \text{FR}^*_\delta\} = \text{MV}_\delta(1),
\]
which leads to (A3). We now prove (A2), which is completed in six different parts.

**Part 1.** Determine \(\phi(\beta_\alpha(S), S)\). The \(\alpha\)- profits for type I and II mergers \(S \in S(k)\) are:

\[
\phi(\beta_\alpha(S), S) = v_\alpha(k) = v^I_\alpha(k) = \frac{(a-c_1)^2(1-2\varepsilon)^2}{(n-k+2)^2}, \quad \text{for type I merger, and}
\]
\[
v_\alpha(k) = v^II_\alpha(k) = \frac{(a-c_1)^2[1+(n-k)\varepsilon]^2}{(n-k+2)^2}, \quad \text{for type II merger, } k = 1, \ldots, n-1. \quad (B23)
\]

**Part 2.** From the above proof for (A3) and the symmetry among inefficient firms, a min solution \(x\) satisfies: \(x_1 = y_1, x_i = y\) all \(i \geq 2\). Using this observation, the constraints in \(\min \{\sum_{i \in \mathcal{N}} x_i \mid x \in \mathbb{R}^n_+ \; ; \sum_{i \in S} x_i \geq v_\alpha(k), \text{ for all } S \in S(k)\}\) can be simplified as:

\[
ky \geq v^I_\alpha(k) = \frac{(a-c_1)^2(1-2\varepsilon)^2}{(n-k+2)^2}, \quad \text{and}
\]
\[
y_1 + (k-1)y \geq v^II_\alpha(k) = \frac{(a-c_1)^2[1+(n-k)\varepsilon]^2}{(n-k+2)^2}, \quad (B24)
\]
\[
y_1 + (k-1)y \geq v^II_\alpha(k) = \frac{(a-c_1)^2[1+(n-k)\varepsilon]^2}{(n-k+2)^2}, \quad (B25)
\]
and the objective function is
\[ \Sigma_{i \in N} x_i = y_1 + (n-1)y. \]  
(B26)

In the next two steps, we determine the smallest \( y_1 \) and \( y \) within \( FR_{\alpha} = \bigcap_{i=1}^{n-1} FR_{\alpha}(k) \).

**Part 3.** Determine the smallest \( y \) among all type I mergers. By (B24),
\[
y \geq \frac{v_{I\alpha}(k)}{k} = \frac{(a-c_1)^2(1-2\varepsilon)^2}{k(n-k+2)^2}, \quad k = 1, \ldots, n-1.
\]
The right side has the same max point of \( MV_{\alpha}(k) \). By (B6), \( \min \{ y \} \) is given by
\[
y^* = \max \left\{ \frac{v_{I\alpha}(k)}{k} | 1 \leq k \leq n-1 \right\} = \begin{cases} 
\frac{(a-c_1)^2(1-2\varepsilon)^2/(n+1)^2}{k(n-k+2)^2} & \text{if } 2 \leq n \leq 5 \\
\frac{(a-c_1)^2(1-2\varepsilon)^2/[9(n-1)]}{k(n-k+2)^2} & \text{if } 6 \leq n.
\end{cases} \]  
(B27)

**Part 4.** Determine the smallest \( y_1 \) among all type II mergers. By (B25) and (B27), for \( k = 1, \ldots, n-1, \) \( y_1 \) satisfies:
\[
y_1 \geq v_{II\alpha}(k) - (k-1)y^* = (a-c_1)^2 \begin{cases} 
\frac{[1+(n-k)\varepsilon]^2}{(n-k+2)^2} - \frac{(k-1)(1-2\varepsilon)^2}{(n+1)^2} & \text{if } 2 \leq n \leq 5 \\
\frac{[1+(n-k)\varepsilon]^2}{(n-k+2)^2} - \frac{(k-1)(1-2\varepsilon)^2}{9(n-1)} & \text{if } 6 \leq n.
\end{cases}
\]  
(B28)

In order to determine \( \max \{ v_{II\alpha}(k) - (k-1)y^* | 1 \leq k \leq n-1 \} \), consider two cases below:

Case 1. \( 2 \leq n \leq 5 \). Consider \( f(k) \), \( k = 1, \ldots, n-1 \), given by
\[
f(k) = \frac{[1+(n-k)\varepsilon]^2}{(n-k+2)^2} - \frac{(k-1)(1-2\varepsilon)^2}{(n+1)^2}.
\]
By \( f(k)'' > 0 \) (i.e., \( f(k) \) is \( \cup \)-shaped), \( \max f(k) = \max \{ f(1), f(n-1) \} \). Therefore,
\[
\max \{ v_{II\alpha}(k) - (k-1)y^* | 1 \leq k \leq n-1 \} = (a-c_1)^2 \max \{ f(1), f(n-1) \}.  
\]  
(B29)

Define \( d(n) = f(1) - f(n-1) \), which is \( \cap \)-shaped (\( d(n)'' < 0 \)). The two roots of \( d(n) = 0 \) are:
\[
n_0 = 9/(1+4\varepsilon^2) - 4, \quad \text{and} \quad n_1 = 2.  
\]  
(B30)

By \( n_0 \geq 2 \iff \varepsilon \leq 1/8 \) (see part (a) of Figure 6), one has:
\[
\max \{ f(1), f(n-1) \} = \begin{cases} 
f(1) & \text{if } \varepsilon \leq 1/8 \text{ and } n \leq n_0 \\
f(n-1) & \text{if } \varepsilon \leq 1/8 \text{ and } n_0 < n \leq 5; \text{ or } 1/8 < \varepsilon \text{ and } n \leq 5.
\end{cases}
\]  
(B31)
Case 2. $6 \leq n$. Now consider the function $f_1(k)$ given by

$$f_1(k) = \frac{[1+(n-k)\varepsilon]^2}{(n-k-1)^2} \cdot \frac{(k-1)(1-2\varepsilon)^2}{9(n-1)}.$$ 

By $f_1(k) > 0$ (so $f_1(k)$ is $\cup$-shaped), $\max f_1(k) = \max \{f_1(1), f_1(n-1)\}$. Therefore,

$$\max \{v_1^{\alpha}(k)-(k-1)y^* | 1 \leq k \leq n-1\} = (a-c_1)^2 \max \{f_1(1), f_1(n-1)\}. \quad (B32)$$

Define $d_1(n) = f_1(1) - f_1(n-1)$. Then, $d_1(n) = 0$ has three roots:

$$n_1 = \frac{2\varepsilon-1-\sqrt{1+116\varepsilon-92\varepsilon^2}}{12\varepsilon} < 0, \quad n_2 = 2, \quad \text{and} \quad n_3 = \frac{2\varepsilon-1+\sqrt{1+116\varepsilon-92\varepsilon^2}}{12\varepsilon}.$$ 

The sign of $d_1(n)$ is the same as that of $g_1(n) = d_1(n)(n-1)(n+1)^2$, whose shape is shown in part (b) of Figure 6. One can show $1 \leq n_3 \leq 5$. Hence, $d_1(n) < 0$ for all $n \geq 6$. This leads to

$$\max \{f_1(1), f_1(n-1)\} = f_1(n-1). \quad (B33)$$

It follows from (B28)-(B33) that $y_1^* = \max \{v_1^{\alpha}(k)-(k-1)y^* | 1 \leq k \leq n-1\}$

$$= (a-c_1)^2 \begin{cases} 
  f(1) & \text{if } \varepsilon \leq 1/8 \text{ and } n \leq n_0 \\
  f(n-1) & \text{if } 1/8 < \varepsilon \text{ and } n_0 < n \leq 5; \text{ or if } 1/8 < \varepsilon \text{ & } n \leq 5 \\
  f_1(n-1) & \text{if } 6 \leq n.
\end{cases} \quad (B34)$$

**Part 5.** Enlarge $FR_\alpha$. By above arguments, $\min \{ \sum x_i | x \in FR_\alpha \}$ is equivalent to

$$\min \{ y_1+(n-1)y | ky \geq v_1^{\alpha}(k) \text{ and } y_1+(k-1)y \geq v_1^{\alpha}(k) \text{ for all } k\}. \quad (B35)$$

Define the following two feasible sets:
FR\text{new} = \{ (y_1,y) \mid ky \geq v_{\alpha}^1(k) \text{ and } y_1+(k-1)y \geq v_{\alpha}^2(k), k = 1, \ldots, n-1\},

FR_{\text{new}}^* = \{ (y_1,y) \mid y \geq y^* \text{ and } y_1 \geq y^*_1\},

where FR_{\text{new}} is the feasible set in (B35). (B27) and (B34) leads to FR_{\text{new}} \subseteq FR_{\text{new}}^*.

**Part 6.** Determine MNBP\(\alpha\). By (B26)-(B27) and (B34)-(B35),

\[
\min \{ y_1+(n-1)y \mid (y_1,y) \in FR_{\text{new}}^* \} = y^*_1+(n-1)y^* = (a-c_1)^2 \{\begin{array}{ll}
\frac{[1+(n-1)\varepsilon]^2+(n-1)(1-2\varepsilon)^2}{(n+1)^2} & \text{if } \varepsilon \leq 1/8 \text{ and } n \leq n_0 \\
\frac{(1+\varepsilon)^2/9+(1-2\varepsilon)^2/(n+1)^2}{(n+1)^2} & \text{if } 1/8 < \varepsilon \text{ and } n \leq 5; \text{ or } 1/8 < \varepsilon \text{ and } n \leq 5 \\
\frac{(1+\varepsilon)^2/9+(1-2\varepsilon)^2/9(n-1)}{(n+1)^2} & \text{if } 6 \leq n.
\end{array}\}
\]  

(B36)

By the maximality in parts (3) and (4), one can check that the minimum solution for each of the three cases in (B36) is included in FR_{\text{new}}. For example, if \(n \geq 6\), the solutions are:

\[y = (a-c_1)^2(1-2\varepsilon)^2 /[9(n-1)], \text{ and } y_1 = (a-c_1)^2\{(1+\varepsilon)^2/9-(n-2)(1-2\varepsilon)^2/[9(n-1)]\} .\]

Hence,

\[\min \{ \sum x_i \mid x \in FR_{\alpha}\} = \min \{ y_1+(n-1)y \mid (y_1,y) \in FR_{\text{new}}^* \} = y^*_1+(n-1)y^* ,\]

which leads to (A2). This completes the proof of Lemma 1. \textbf{Q.E.D}

**Proof of Proposition 6:** To prove part (i), we need to show \(d = v(N) - \text{MNBP}_{\alpha} > 0\). Consider each of the three cases in (A2). Case 1. \(\varepsilon \leq 1/8\) and \(n \leq n_0 (\leq 5)\). \(d\) is given by

\[d = (a-c_1)^2\left\{\frac{1}{4} - \frac{[1+(n-1)\varepsilon]^2+(n-2)(1-2\varepsilon)^2}{(n+1)^2}\right\} = d_1(n) / (n+1)^2 .\]

where \(d_1(n) = 0\) has two roots: \(n_1 < n_2 = 1\). By \(d_1(n)^\nu < 0\), \(d > 0\) for all \(n \geq 2\).

Case 2. \(\varepsilon \leq 1/8\) and \(n_0 < n \leq 5\); or \(1/8 < \varepsilon\) and \(n \leq 5\). \(d\) is given by

\[d = d_2(n, \varepsilon) = (a-c_1)^2\left\{\frac{1}{4} - \frac{(1+\varepsilon)^2}{9} \frac{(1-2\varepsilon)^2}{(n+1)^2}\right\},\]

By \(d_2(2, 0) > 0\), \(d_2(2, 0.5) = 0\), and \(d_2(2, \varepsilon)^\nu < 0\), \(d_2(2, \varepsilon) \geq 0\) for all \(\varepsilon\). By \(d_2(n)^\nu > 0\), \(d_2(3, \varepsilon), d_2(4, \varepsilon), d_2(5, \varepsilon) > 0\) for all \(\varepsilon \in [0, 0.5]\). Hence, \(d > 0\) for all \(2 \leq n \leq 5\).

Case 3. \(6 \leq n\). \(d\) is given by

\[d = d_3(n) = (a-c_1)^2\left\{\frac{1}{4} - \frac{(1+\varepsilon)^2}{9} \frac{(1-2\varepsilon)^2}{9(n-1)^2}\right\},\]

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By \(d_3(n)' > 0\), \(d_3(n)\) is increasing. By \(d_3(6) > 0\), \(d > 0\) for all \(6 \leq n\). This proves part (i).

Now we prove parts (ii) and (iii). \(d = v(N) - \text{MNBP}_\delta\) is given by

\[
d = d_4(n) = \frac{(a-c_1)^2}{4} \cdot \left( 1 + \frac{(1+\epsilon)^2}{2} \right) \cdot \left( \frac{(1-\epsilon)^2}{8} \right) \cdot \left( \frac{1}{2-\epsilon} \right) \cdot \left( \epsilon \frac{4n-9}{8n-6} \right).
\]

By \(\epsilon \leq 1/2\), \(d > 0\) if and only if \(\epsilon \geq \omega_0 = (4n-9)/(8n-6)\). Q.E.D

The following expressions are used in proofs for Lemmas 2-3 and Propositions 2-4, where \(\tilde{\pi}_{12}(\Delta_3) = v_{13}, \tilde{\pi}_{13}(\Delta_2) = v_{13}, \text{ and } \tilde{\pi}_{23}(\Delta_1) = v_{23}\) (see (18) or (A10)).

\[
v_1^a = \pi_1 = (a-c_1)^2 \left( 1 + 2\epsilon + \epsilon^2 \right)/16, \quad v_2^a = \pi_2 = (a-c_1)^2 \left( 1 - 3\epsilon^2 + \epsilon^3 \right)/16,
\]

\[
v_3^a = \pi_3 = (a-c_1)^2 \left( 1 - 3\epsilon^3 + \epsilon^2 \right)/16;
\]

\[
v_1^\delta = (a-c_1)^2 \left( 1 + \epsilon^2 \right)/9, \quad v_2^\delta = (a-c_1)^2 \left( 1 - 2\epsilon^2 \right)/9, \quad v_3^\delta = (a-c_1)^2 \left( 1 - 2\epsilon^3 \right)/9;
\]

(C1)

Proof of Lemma 2: Consider first \(S = 12\). Let \(d_{12} = \tilde{\pi}_{12}(\Delta_3) - (\pi_1 + \pi_2)\). (C1) leads to

\[
d_{12}(\epsilon) = (a-c_1)^2 (1 + \epsilon^3 - 3\epsilon^2)(15\epsilon^2 - 1 - \epsilon^3)/72.
\]

By A0, \((1+\epsilon^2 - 3\epsilon^3) > 0\). Hence, \(d_{12} > 0 \iff \epsilon < \theta_2\), which is given by

\[
\theta_2 = 15\epsilon^2 - 1. \quad (C2)
\]

Now consider \(S = 13\). Let \(d_{13} = \tilde{\pi}_{13}(\Delta_2) - (\pi_1 + \pi_3)\). (C1) leads to

\[
d_{13}(\epsilon) = (a-c_1)^2 (1 + \epsilon^3 - 3\epsilon^2)(15\epsilon^3 - 1 - \epsilon^2)/72.
\]

By \((1+\epsilon^2 - 3\epsilon^3) > 0\), \(d_{13} > 0 \iff \epsilon > \theta_4\), which is given by

\[
\theta_4 = (1+\epsilon^2)/5. \quad (C3)
\]

Finally, consider \(S = 23\). Let \(d_{23} = \tilde{\pi}_{23}(\Delta_1) - (\pi_2 + \pi_3)\). By (C1), one has

\[
d_{23}(\epsilon) = \frac{5(a-c_1)^2}{8} \left( \frac{1+\epsilon^2}{3} + \epsilon_3^3(\epsilon_3^2 - \frac{1+13\epsilon_3}{15}) \right) = \frac{5(a-c_1)^2}{8} (\theta_0 - \epsilon_3)(\epsilon_3^2 - \theta_6).
\]

By \((\theta_0 - \epsilon_3) > 0\), \(d_{23} > 0 \iff \epsilon_3 > \theta_6\), where \(\theta_0\) and \(\theta_6\) are given by

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\[
\theta_0 = (1+\varepsilon_2)/3; \quad \theta_6 = (1+13\varepsilon_2)/15.
\] (C4)

This completes the proof for Lemma 2. \hspace{1cm} \textbf{Q.E.D}

The following relations (see Figure 5) are useful in proving Lemma 3.

\[
\varepsilon_3 \geq \theta_2 \text{ if } \varepsilon_2 \leq 1/14; \quad \varepsilon_3 < \theta_2 = 15\varepsilon_2-1 \text{ if } \varepsilon_2 > 1/11;\] (C5)

\[
\varepsilon_3 > \theta_4 = (1+\varepsilon_2)/5 \text{ if } \varepsilon_2 > 1/14;\] (C6)

\[
\theta_4 < \theta_6 \leq \theta_0; \quad \varepsilon_2 \leq \varepsilon_3 \leq \theta_0; \text{ and } \varepsilon_2 \leq \theta_6.\] (C7)

\textbf{Proof of Lemma 3:} We first compute \(\text{MNBP}_\delta\). There are six constraints: \(x_1 \geq v_{1}\), \(x_2 \geq v_{2}\), \(x_3 \geq v_{3}\); \(x_1+x_2 \geq v_{12}\), \(x_1+x_3 \geq v_{13}\), \(x_2+x_3 \geq v_{23}\). By \(v_{\delta} = v_{12}\), \(v_{\delta} = v_{23}\), the problem becomes:

\[
\text{MNBP}_\delta = \text{Min } \{ x_1+x_2+ x_3 | x_1 \geq v_{1}, x_2 \geq v_{2}, x_3 \geq v_{3}; x_1+x_2 \geq v_{12}\},\] (C8)

of which the minimum value is equal to

\[
v_{3}^\delta + \text{Max } \{v_{1}^\delta + v_{2}^\delta, v_{3}\}.
\] (C9)

Let

\[
d(\varepsilon_3) = v_{1}^\delta + v_{2}^\delta - v_{12} = (a-c_1^2)((1+\varepsilon_2)^2+(1-2\varepsilon_2)^2-(1+\varepsilon_3)^2)/9.
\]

By \(d^{''} < 0\), \(d\) is \(\cap\)-shaped. \(d(\varepsilon_3) = 0\) has two roots: \(\mu_1 < 0 < \mu_2\), where \(\mu_2\) is given by

\[
\mu_2 = \rho_0 = -1+ \sqrt{2-2\varepsilon_2+5\varepsilon_3^2}.
\]

Hence, \(\text{Max } \{v_{1}^\delta + v_{2}^\delta, v_{3}\} = v_{1}^\delta + v_{2}^\delta\) if \(\varepsilon_3 \leq \rho_0\), \(v_{3}\) if \(\varepsilon_3 > \rho_0\). Then, (C1) and (C8)-(C9) lead to (A12).

We only provide an outline for proving (A13)-(A16), because complete proofs like those for (A12) would make the paper too long. Figure 5 illustrates all the sub-cases.

Case 1. \(\varepsilon_2 \in [0, \frac{1}{14}]\). By Lemma 2, \(d_{12} \leq 0\), so only five constraints are left:

\[
x_1 \geq v_1 = v_1^\alpha, \quad x_2 = v_2 = v_2^\delta, \quad x_3 = v_3 = v_3^\delta; \quad x_1+x_3 \geq v_{13}. \quad x_2+x_3 \geq v_{23}.
\]

Let \(h_1 = v_{13}-v_1\), \(h_2 = v_{23}-v_2\), one has \(d(\varepsilon_2, \varepsilon_3) = \text{max } \{h_1, h_2\} = h_1\), and \(v_3 \geq d(\varepsilon_2, \varepsilon_3) \Leftrightarrow \varepsilon_3 \leq \theta_4\).

By \(\text{MNBP}_\alpha = v_{1} + v_{2} + \max \{v_{3}, d(\varepsilon_2, \varepsilon_3)\}\), \(\text{MNBP}_\alpha = v_{1} + v_{2} + v_{3}\), if \(\varepsilon_3 < \theta_4\), and \(\text{MNBP}_\alpha = v_{2} + v_{13}\) if \(\varepsilon_3 \geq \theta_4\). This proves (A13).

Case 2. \(\varepsilon_2 \in [\frac{1}{14}, \frac{4}{53}]\). One has \(\theta_4 \leq \varepsilon_2 \leq \theta_2 < \theta_6\). If \(\varepsilon_3 \geq \theta_2\), then \(d_{12} \leq 0\). By Case 1,
\( \text{MNBP}_\alpha = v_2 + v_{13} \). If \( \varepsilon_3 < \theta_2 < \theta_6 \), then \( d_{23} \leq 0 \). So the constraint \( x_2 + x_3 \geq v_{23} \) can be removed.

Using similar steps as in Case 1, one can show \( \text{MNBP}_\alpha = v_2 + v_{13} \).

Case 3. \( \varepsilon_2 \geq \frac{4}{53} \), and \( \varepsilon_3 \leq \theta_6 \). By \( d_{23} \leq 0 \), \( x_2 + x_3 \geq v_{23} \) is removed. Similar to Case 2, and by \( d_{13} > 0 \), one can show \( \text{MNBP}_\alpha = v_2 + v_{13} \) if \( \varepsilon_3 \leq \rho_1 \), and = \( v_3 + v_{12} \) if \( \varepsilon_3 > \rho_1 \). One can also show that \( \frac{4}{53} \leq \varepsilon_2 \leq \frac{16}{77} \) implies \( \varepsilon_3 \leq \rho_1 \), and \( \varepsilon_2 \geq \frac{5}{22} \) implies \( \varepsilon_3 > \rho_1 \).

Case 4. \( \varepsilon_2 \geq \frac{4}{53} \), and \( \varepsilon_3 \geq \theta_2 > \theta_6 \). This can only occur for \( \varepsilon_2 \in \left[ \frac{4}{53}, \frac{1}{11} \right] \). By Case 1, \( d_{12} \leq 0 \), and \( \varepsilon_3 \geq \theta_4 \), \( \text{MNBP}_\alpha = v_2 + v_{13} \). By Cases 2-3, one gets (A14) and (A15).

Case 5. \( \varepsilon_2 \geq \frac{4}{53} \), and \( \theta_2 \geq \varepsilon_3 > \theta_6 \). One has \( d_{12} > 0 \), \( d_{13} > 0 \), \( d_{23} > 0 \). Note at most one of \( x_1 \geq v_1 \), \( x_2 \geq v_2 \), \( x_3 \geq v_3 \) can be binding. First solving each of the three cases: Case 5.1, \( x_1 = v_1 \); Case 5.2, \( x_2 = v_2 \); Case 5.3, \( x_3 = v_3 \). Now solve Case 5.4, \( x_1 > v_1 \), \( x_2 > v_2 \), \( x_3 > v_3 \). In case 5.4, one must have \( x_1 + x_2 = v_{12} \), \( x_1 + x_3 = v_{13} \), and \( x_2 + x_3 = v_{23} \). Solving these equations, one gets \( y_1, y_2, y_3 \). By checking \( y_i > v_i \), and using Cases 5.1-5.3, one can get (A16). \( \Box \) \( \Box \)

**Proof of Proposition 2**: Part (i). For each of the values of \( \text{MNBP}_\alpha \), one can show \( v(N) > \text{MNBP}_\delta \). Now consider part (ii). If \( \varepsilon_3 \leq \rho_0 \), \( d = v(N) - \text{MNBP}_\delta \) is given by

\[
d(\varepsilon_3) = (a-c_1)^2 \left[ \frac{1}{4} - \frac{(1+\varepsilon_2)^2 + (1-\varepsilon_2)^2}{9} \right].
\]

Using \( d'' < 0 \) and solving the two roots \( \mu_1 < \mu_2 \) for \( d(\varepsilon_3) = 0 \), one has:

\[
d > 0 \iff \omega_1 \leq \varepsilon_3 \leq \rho_0, \text{ where } \omega_1 = \frac{1}{2} \sqrt{\frac{1+8\varepsilon_2-20\varepsilon_2^2}{4} - \rho_0}. \tag{C10}
\]

If \( \varepsilon_3 > \rho_0 \), \( d \) is given by

\[
d(\varepsilon_3) = (a-c_1)^2 \left[ \frac{1}{4} - \frac{(1+\varepsilon_3)^2 + (1-\varepsilon_3)^2}{9} \right] = (a-c_1)^2 \left[ \frac{1}{36} - (1+10\varepsilon_3)/(1-2\varepsilon_3) \right] > 0.
\]

Using (C10), one gets \( d > 0 \iff \omega_1 \leq \varepsilon_3 \), which completes the proof of part (ii). \( \Box \) \( \Box \)

**Proof of Proposition 3**: Since \( \Delta_1 = \{1; 23\} \) and \( \Delta_2 = \{13; 2\} \) have identical welfare, \( \Delta_2 \) can be ignored. First, evaluate six cases below. (1) \( \Delta_0 \rightarrow \Delta_3 \). Let \( d_1(\varepsilon_3) = W_3 - W_0 \). Using \( d_1(\varepsilon_3)'' < 0 \) and solving \( d_1 = 0 \), one can show: \( d_1(\varepsilon_3) \geq 0 \iff \varepsilon_3 \leq \sigma_1 = (-7+69\varepsilon_2)/31, \varepsilon_3 < \sigma_1 \) if \( \varepsilon_2 > 13/44 \), and \( \varepsilon_3 > \sigma_1 \) if \( \varepsilon_2 < 7/38 \);

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(2) $\Delta_0 \rightarrow \Delta_1$. $d_4(\varepsilon_3) = W_2 - W_0$ leads to: $d_4(\varepsilon_3) \geq 0 \iff \varepsilon_3 \geq \sigma_2$; $\varepsilon_3 > \sigma_2$ if $\varepsilon_2 > 7/38$, where

$$\sigma_2 = \omega_2 = (7+31\varepsilon_2)/69; \quad (C11)$$

(3) $\Delta_3 \rightarrow \Delta_1$. $d_4(\varepsilon_3) = W_1 - W_3$ leads to: $d_4(\varepsilon_3) \geq 0 \iff \varepsilon_3 \geq \sigma_3 = (-\varepsilon_2 + 8/11)$, $\varepsilon_3 \leq \theta_0 < \sigma_3$ if $\varepsilon_2 < 13/44$, and $\varepsilon_3 \geq \varepsilon_2 > \sigma_3$ if $\varepsilon_2 > 4/11$;

(4) $\Delta_1 \rightarrow \Delta_m$. $d_4(\varepsilon_3) = W_m - W_1$ leads to: $d_4(\varepsilon_3) \geq 0 \iff \varepsilon_2 \geq \sigma_4 = 5/22$;

(5) $\Delta_0 \rightarrow \Delta_m$. $d_4(\varepsilon_3) = W_m - W_0$ leads to: $d_4(\varepsilon_3) \geq 0 \iff \sigma_5 \leq \varepsilon_3 \leq \sigma_6$, $d_5 < 0$ if $\varepsilon_2 < \sigma_7 = 5/14 - \sqrt{23}/28 \approx 0.19$, and $d_5 > 0$ if $\varepsilon_2 > 5/22$, where $\sigma_5$ and $\sigma_6$ are given by

$$\sigma_5 = (5+9\varepsilon_2 - 23/28) = (7+9\varepsilon_2 + 23/28)/23, \quad \sigma_6 = (5+9\varepsilon_2 - 23/28) = (7+9\varepsilon_2 + 23/28)/23; \quad (C12)$$

(6) $\Delta_3 \rightarrow \Delta_m$. $d_4(\varepsilon_3) = W_m - W_3$ leads to $d_4(\varepsilon_3) \geq 0 \iff \varepsilon_3 \geq 5/22$.

Second, comparing cases 1-6 on $[0, 0.5]$ and picking up the maximal $W$, one gets: $W^* = W_m$ if $\varepsilon_2 > 5/22$; $W_1$ if $7/38 < \varepsilon_2 \leq 5/22$; $W_1$ if $\varepsilon_2 \leq 7/38$ and $\varepsilon_3 \geq \omega_2$; $W_0$ if $\varepsilon_2 \leq 7/38$ and $\varepsilon_3 < \omega_2$.

**Q.E.D**

**Proof of Proposition 4:** Part (i) Consider the stability of $\Delta_1 = \{1; 23\}$ with $y$ given by $y_1 = \nu_1^\delta = \nu_{1s}$, $y_2 = \nu_2 + t d_{23}$, and $y_3 = \nu_3 + (1-t)d_{23}$. By $d_{23} > 0$ (i.e., $\varepsilon_3 > \theta_0$) and the definition of $y$, $y \in Y(\Delta_1) = \{y \mid y \geq \nu_1^\delta = \nu_1, y \geq \nu_2, y \geq \nu_3, y + y \geq \nu_1, y + y \geq \nu_2, y + y \geq \nu_3\} = Y_0(\Delta_1) = Y_\delta(\Delta_1)$ is equivalent to

$$d(\varepsilon_3) = \nu_0 + \nu_2 + t d_{23} \nu_{12} \geq 0. \quad (C13)$$

Note $d'' < 0$ (i.e., $d$ is $\cap$-shaped), and $d(\varepsilon_3) = 0$ has two roots:

$$\mu_1(\varepsilon_2, t) = \frac{-14 - 54\varepsilon_2 + 36(1+3\varepsilon_2) + 8 \sqrt{7+14\varepsilon_2 + 88\varepsilon_2^2 + t(34-244\varepsilon_2 + 352\varepsilon_2^2) + t^2(1-4\varepsilon_2 + 4\varepsilon_2^2)}}{2(7+90t)}, \quad (C14)$$

$$\mu_{10}(\varepsilon_2, t) = \frac{-14 - 54\varepsilon_2 + 36(1+3\varepsilon_2) - 8 \sqrt{7+14\varepsilon_2 + 88\varepsilon_2^2 + t(34-244\varepsilon_2 + 352\varepsilon_2^2) + t^2(1-4\varepsilon_2 + 4\varepsilon_2^2)}}{2(7+90t)}.$$  

It can be checked that the following three claims hold:

$$\mu_{10}(\varepsilon_2, t) < \theta_0; \quad \theta_0 \leq \mu_1(\varepsilon_2, t) \iff \varepsilon_2 \leq 1/11; \quad \text{and} \quad \theta_0 \leq \mu_1(\varepsilon_2, t) \iff \varepsilon_2 \leq 113/316. \quad (C15)$$

Therefore, by (C15), by the $\cap$-shape of $d(\varepsilon_3)$, and by $\theta_0 \leq \varepsilon_3 \leq \theta_0$, one has
\[ d(\varepsilon_3) = \begin{cases} 
> 0 & \text{if } \varepsilon_2 \leq 1/11 \\
\geq 0 & \varepsilon_3 \leq \mu_1(\varepsilon_2; t) \text{ if } 1/11 < \varepsilon_2 < 113/316 \\
< 0 & \text{if } \varepsilon_2 \geq 113/316; 
\end{cases} \]

which leads to part (i).

One can double check the above results by evaluating \( t = 1 \) and \( t = 0 \) separately, the results in these separate cases are the same as those by replacing the above \( t \) by 0 and 1.

The proofs for parts (ii)-(iii) are similar. In particular, \( \mu_2(\varepsilon_2, t) \) for \( \Delta_2 \) in part (ii) is:

\[
\mu_2(\varepsilon_2, t) = \frac{-14+18\varepsilon_2+36t(1+\varepsilon_2)+8\sqrt{7-28\varepsilon_2+37\varepsilon_2^2+t(34-256\varepsilon_2+430\varepsilon_2^2)+9t^2(1+2\varepsilon_2+\varepsilon_2^2)^2}}{2(7+90t)}. \tag{C16}
\]

Note the formula for (iii) is given by that for (ii), after switching \( \varepsilon_2 \) and \( \varepsilon_3 \).

**Q.E.D**

**REFERENCES**


