MULTIVARIABLE FUZZY CONTROLLER UNDER GÖDEL'S IMPLICATION

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Abstract: In this paper, the upper bounds for a multivariable fuzzy logic controller under Gödel's implication are investigated and a generalized multivariable structure of fuzzy controller are proposed. Contribution operators, such as max (\( \vee \)) operation, ensure the greatest upper bound as well as the least upper bound. Conditions which determine an appropriate upper bound are also determined. This theory has applications in the design of fuzzy logic controllers.

Keywords: Upper bound; multivariable fuzzy controller; Gödel's implication; fuzzy relational equations; approximate reasoning; fuzzy logic; fuzzy systems; fuzzy sets.

1. Introduction

The main purpose of this paper is to establish the upper bounds for the outputs of multivariable fuzzy controllers under Gödel's implication.

A number of studies [1–6] have been carried out on the design of fuzzy logic controllers for dynamic processes since the publications of L.A. Zadeh in 1973 [7] and E.H. Mamdani in 1974 [8]. Recently, some studies have been reported on multidimensional fuzzy reasoning [9, 10] and multivariable fuzzy controllers [11, 12]. Shakouir et al. [13] proposed a fuzzy control algorithm for multivariable systems which is based on a state-space model of the system. Cheng et al. [14] suggested intersection coefficients and a formula for a multivariable fuzzy controller in terms of these coefficients. Czogala and Zimmermann [15] introduced random-intersection coefficients and a structure for a multidimensional probabilistic fuzzy controller using these coefficients.

Walichiewicz [16] proposed a multidimensional fuzzy controller using the

Gupta, Kiszka and Trojan [11] used a concept called the decomposition of multivariable control rules, fuzzy relations and intersection operators to model the multivariable structure of a fuzzy system. Extended fuzzy reasoning forms with several fuzzy conditional propositions, combined with an 'else' connective, have been extensively studied by Mizumoto [9, 21]. Mizumoto found that the consequence \( C' \) is not equal to but contained in the intersection of each fuzzy inference result \( (A' \text{ and } B') \circ (A_i \text{ and } B_i \rightarrow C_i) \). A multivariable fuzzy controller for a two-tank liquid-level system has been constructed and analyzed by Gupta, Kiszka and Trojan [11, 12]. It was found that the real values of the present output sets are upper bounded by the calculated values of these outputs [11].

The results described above motivated Kiszka, Gupta and Trojan [22] to analyse upper bounds for outputs of multivariable fuzzy controllers. Gödel's fuzzy implication is used extensively by many designers to build fuzzy controllers and to model fuzzy systems [9, 20, 23].

It is found in our studies that Gödel's implication secures the least root-mean-square error with a minimal number of mathematical operations necessary for implementation of the fuzzy controllers [23]. It has been reported also that Gödel's implication secures the greatest solution of fuzzy relational equations [19, 20].

In this paper, we investigate the upper bounds for outputs of multivariable fuzzy controllers under Gödel's fuzzy implication. The generalized multivariable structures of fuzzy controllers are discussed in Section 2. Such structures incorporate translating and inferring stages of knowledge-base processing, and combine fuzzy rules with a generalized inference mechanism into a unified form of multivariable fuzzy equations. Composition operators, contribution operators, aggregation operators, implication operators, multivariable fuzzy rules, and current inputs and outputs are major components of such multivariable structures.

Upper bounds for present outputs of a multivariable fuzzy controller are analysed in detail in Section 3. Two categories of upper bounds have been identified. The greatest upper bound and the least upper bound were found. The contribution operator which assures the above bounds is also investigated in Section 3. Several numerical examples are presented to illustrate the use of the theory.

This paper shows that the system designer has a choice of an appropriate theoretical structure for a multivariable fuzzy controller so as to obtain the desirable features of the present fuzzy outputs. This avoids expensive simulation studies based on upon trial-and-error calculations.
2. Multivariable fuzzy controller

In general, a multivariable linguistic description of a fuzzy controller is given by [9, 11]

\[
\begin{align*}
\text{if } A^{(i)}(u) \text{ and } B^{(i)}(v) \text{ then } C^{(i)}(w) \text{ and } D^{(i)}(z) \\
\text{also} \\
\text{also} \\
\text{if } A^{(j)}(u) \text{ and } B^{(j)}(v) \text{ then } C^{(j)}(w) \text{ and } D^{(j)}(z)
\end{align*}
\]

(1)

and \(A^{(i)}(u)\) and \(B^{(i)}(v)\) are the fuzzy values of \(A\)-input and \(B\)-input variables defined in the universe of discourses \(U\) and \(V\) respectively; \(C^{(i)}(w)\) and \(D^{(i)}(z)\) are fuzzy values of \(C\)-output and \(D\)-output variables defined in the universe of discourses \(W\), and \(Z\), respectively; and \(i = 1, 2, \ldots, N\), are the number of fuzzy rules. This controller has two inputs and two outputs and is depicted in Figure 1.

Let the dimensions of the universes of discourse \(U\), \(V\), \(W\) and \(Z\) be \(\dim U = q_1\), \(\dim V = q_2\), \(\dim W = p_1\) and \(\dim Z = p_2\) respectively. Let the basic families of input fuzzy sets and the output fuzzy sets be given by

\[
\begin{align*}
\mathcal{A} &= \{A \mid U \ni A_i \rightarrow [0, 1], i = 1, 2, \ldots, N\}, \\
\mathcal{B} &= \{B \mid V \ni B_i \rightarrow [0, 1], i = 1, 2, \ldots, N\}, \\
\mathcal{C} &= \{C \mid W \ni C_i \rightarrow [0, 1], i = 1, 2, \ldots, N\}, \\
\mathcal{D} &= \{D \mid Z \ni D_i \rightarrow [0, 1], i = 1, 2, \ldots, N\}.
\end{align*}
\]

To obtain the present outputs \(C'(w)\), \(D'(z)\), given the current inputs \(A'(u)\), \(B'(v)\), the following form of multivariable fuzzy inference is proposed [22]:

\[
\begin{align*}
C'(w) &= A'(u) \ast R_{11}(u, w) \Delta B'(v) \ast R_{12}(v, w), \\
D'(z) &= A'(u) \ast R_{21}(u, z) \Delta B'(v) \ast R_{22}(v, z),
\end{align*}
\]

(2)

where \(\ast\) denotes a certain composition, and \(\Delta\) is a contribution operator.

![Multivariable Fuzzy System](image)

Fig. 1. Multi-input – multi-output fuzzy system.
The two-dimensional fuzzy relations are defined as:

\[ R_{11} : A \times C \rightarrow [0, 1], \]
\[ R_{12} : B \times C \rightarrow [0, 1], \]
\[ R_{21} : A \times D \rightarrow [0, 1], \]
\[ R_{22} : B \times D \rightarrow [0, 1], \]

where \( \times \) stands for the Cartesian product.

A block diagram showing the multivariable structure of the proposed fuzzy controller (2) is shown in Figure 2. This structure consists of the current input signals, the branch points, the functional blocks, the inner signals, the contributions blocks, and the present output signals.

A generalized form of the fuzzy relations is given by [22].

\[ R_{11}(u, w) = \bigcirc_i \{ A^{(i)}(u) \bigcirc \ C^{(i)}(w) \}, \]
\[ R_{12}(v, w) = \bigcirc_i \{ B^{(i)}(v) \bigcirc \ C^{(i)}(w) \}, \]
\[ R_{21}(u, z) = \bigcirc_i \{ A^{(i)}(u) \bigcirc \ D^{(i)}(z) \}, \]
\[ R_{22}(v, z) = \bigcirc_i \{ B^{(i)}(v) \bigcirc \ D^{(i)}(z) \}, \]

where \( i = 1, 2, \ldots, N \); \( \bigcirc_i \) is a generalized aggregation operator, and \( \bigcirc \) is a generalized implication operator. Also, \( \dim R_{11} = q_1 \times p_1 \), \( \dim R_{12} = q_2 \times p_1 \), \( \dim R_{21} = q_1 \times p_2 \), and \( \dim R_{22} = q_2 \times p_2 \).

Substituting Eqs. (3) into (2), we obtain the following form of generalized
multivariable fuzzy reasoning:
\[ C'(w) = A'(u) \ast \{\bowtie \ [A^{(i)}(u) \circ C^{(i)}(w)]\} \]
\[ \Delta B'(v) \ast \{\bowtie \ [B^{(i)}(v) \circ C^{(i)}(w)]\}, \]  
\[ D'(z) = A'(u) \ast \{\bowtie \ [A^{(i)}(u) \circ D^{(i)}(z)]\} \]
\[ \Delta B'(v) \ast \{\bowtie \ [B^{(i)}(v) \circ D^{(i)}(z)]\}. \]  

Eqs. (4) reflects the following two stages of multivariable approximate reasoning:
(i) Translating stage: to get fuzzy relations from a set of fuzzy rules;
(ii) Inferring stage: to find the the present outputs from the current inputs based upon the available fuzzy relations.

It is worth noting that the \( \bowtie \) and \( \circ \) operators are responsible for the translating stage, and that the \( \ast \) and \( \Delta \) operators account for the inferring stage. Eqs. (4) combine the fuzzy rules (1), as well as a generalized inference mechanism, in a unified form of fuzzy equations. Therefore, the set of equations in (4) is very convenient for its use in the investigation of the properties of multivariable fuzzy controllers. The generalized implication operator \( \bowtie \), the contribution operator \( \Delta \) and an aggregation operator \( \circ \) are two phase functions from \([0, 1] \times [0, 1]\) to \([0, 1]\). Therefore, in general, they can be expressed using the notion of t-norm and t-conorm [19, 20].

A function \( T: [0, 1] \times [0, 1] \rightarrow [0, 1] \) is called a t-norm if it fulfills the following conditions:
(1) Boundary condition:
\[ T(0, 0) = 0, \quad T(u, 1) = u. \]
(2) Monotonicity:
\[ T(u, v) \leq T(w, z) \quad \text{for} \quad u \leq w, v \leq z. \]
(3) Symmetry:
\[ T(u, v) = T(v, u). \]
(4) Associativity:
\[ T(T(u, v), w) = T(u, T(v, w)). \]

If \( T \) is a t-norm then the function \( S: [0, 1] \times [0, 1] \rightarrow [0, 1] \) defined by
\[ S(u, v) = 1 - T(1 - u, 1 - v) \]
is called its t-conorm.

A composition operator \( \ast \) can be defined by sup-t operation [20]. If \( \ast = \max - \min, \quad \bowtie = \max_i, \quad \circ = \min, \) and \( \Delta = \max, \) the following form of
Mamdani’s multivariable fuzzy controller can be obtained using Eqs. (4):

\[ C'(w) = \bigvee_u \left\{ A'(u) \land \bigvee_i [A^{(i)}(u) \land C^{(i)}(w)] \right\} \land \bigvee_v \left\{ B'(v) \land \bigvee_i [B^{(i)}(v) \land C^{(i)}(w)] \right\}, \]  

\[ (5a) \]

\[ D'(v) = \bigvee_u \left\{ A'(u) \land \bigvee_i [A^{(i)}(u) \land D^{(i)}(z)] \right\} \land \bigvee_v \left\{ B'(v) \land \bigvee_i [B^{(i)}(v) \land D^{(i)}(z)] \right\} \]  

where \( \lor \) and \( \land \) denote, respectively, max and min operators, and \( \bigvee_u \) and \( \bigvee_v \) are the supremum over the domains of \( u \) and \( v \).

Eqs. (5) have been studied by Gupta, Kiszka and Trojan in [11, 12 and 22].

3. Upper bounds for Gödel’s multivariable fuzzy controller

The basic definition of an upper bound of an output fuzzy set has the following form [18]: An upper bounded fuzzy subset \( \tilde{A} \) of \( U \) is a collection of fuzzy subsets

\[ \tilde{A} = \{ A \mid A \subseteq A_{up}, \forall A \subseteq U \}, \]  

\[ (6) \]

where \( A_{up} \) is called the upper bound of \( A \). An upper bound fuzzy subset is shown in Figure 3. The following definition will be used in further investigations of the upper bound [7]. The height of a fuzzy set \( A \) is the largest membership grade of any element in \( A \). It is denoted as \( \text{hgt}(A) \) and is defined as

\[ \text{hgt}(A) = \bigvee_{u \in U} A(u). \]

The problem of concern here is to find an upper bound for the present output fuzzy sets \( C' \) and \( D' \), given the current input fuzzy sets \( A' \) and \( B' \) in a multivariable structure of the fuzzy controller defined in (4). In this respect, the aggregation operator \( \ominus \) and the contribution operator \( \Delta \) can take different

![Fig. 3. Upper bounded fuzzy subsets.](image-url)
appropriate forms. The composition operator $*$ is defined by the max–min operation and the implication operator $\circ$ is expressed by Gödel’s implication $[9, 23]:$

$$\circ = a \rightarrow b = \begin{cases} 1, & a \leq b, \\ b, & a > b. \end{cases}$$ (7)

We will now investigate four different cases of these operators using the appropriate propositions and examples.

**Case (i)**

For this case, let us consider the following definitions of the operators:
- Aggregation operator: $\mathcal{G} = \bigvee.$
- Implication operator: $\circ$ is Gödel's implication as defined in (7).
- Contribution operator: $\Delta = \wedge.$

The present output of fuzzy sets can now be obtained using Eqs. (4) and is given by:

$$C'(w) = \bigvee\left[ A'(u) \wedge \bigvee_i \left\{ \begin{array}{l} 1, \\ A^{(i)}(u) \leq C^{(i)}(w) \end{array} \right. \right]$$

Thus, we have

$$C'(w) = \left\{ \begin{array}{l} \bigvee u A'(u) \wedge \bigvee v B'(v), \\ A^{(i)}(u) \wedge B^{(i)}(v) \leq C^{(i)}(w), \\ \bigvee u A'(u) \wedge \bigvee v B'(v) \wedge \bigvee i C^{(i)}(w), \\ A^{(i)}(u) \wedge B^{(i)}(v) > C^{(i)}(w). \end{array} \right. $$ (8)

Also,

$$D'(z) = \bigvee\left[ A'(u) \wedge \bigvee_i \left\{ \begin{array}{l} 1, \\ A^{(i)}(u) \leq D^{(i)}(z) \end{array} \right. \right]$$

Hence it follows that

$$D'(z) = \left\{ \begin{array}{l} \bigvee u A'(u) \wedge \bigvee v B'(v), \\ A^{(i)}(u) \wedge B^{(i)}(v) \leq D^{(i)}(z), \\ \bigvee u A'(u) \wedge \bigvee v B'(v) \wedge \bigvee i D^{(i)}(z), \\ A^{(i)}(u) \wedge B^{(i)}(v) > D^{(i)}(z). \end{array} \right. $$ (9)

In view of Eqs. (8) and (9), we state the following proposition:
Proposition 1. (1a) If

\[ \left( \bigvee_u A^{(i)}(u) \land \bigvee_v B^{(i)}(v) \right) \leq \bigvee_w C^{(i)}(w) \]

and

\[ \left( \bigvee_u A^{(i)}(u) \land \bigvee_v B^{(i)}(v) \right) \leq \bigvee_z D^{(i)}(z), \]

then

\[ C'(w) \subseteq \left( \bigvee_u A'(u) \land \bigvee_v B'(v) \right) \]

and

\[ D'(z) \subseteq \left( \bigvee_u A'(u) \land \bigvee_v B'(v) \right). \] (10)

In other words, if the least height of one of the entries pair of the basic family of input sets is equal to or less than the height of one of the entries of the basic family of output sets, then the height of the present output sets is upper bounded by the least height of the current input sets.

(1b) If

\[ \left( \bigvee_u A^{(i)}(u) \land \bigvee_v B^{(i)}(v) \right) > \bigvee_w C^{(i)}(w), \]

\[ \left( \bigvee_u A'(u) \land \bigvee_v B'(v) \right) \geq \bigvee_i C^{(i)}(w) \] (11a)

and

\[ \left( \bigvee_u A^{(i)}(u) \land \bigvee_v B^{(i)}(v) \right) > \bigvee_z D^{(i)}(z), \]

\[ \left( \bigvee_u A'(u) \land \bigvee_v B'(v) \right) \geq \bigvee_i D^{(i)}(z), \] (11b)

then

\[ C'(w) \subseteq \bigvee_i C^{(i)}(w) \quad \text{and} \quad D'(z) \subseteq \bigvee_i D^{(i)}(z). \]

That is, if the least height of one of the entries pair of the basic family of input sets is greater than the height of one of the entries of the basic family of output sets and the least height of the current input sets is equal to or greater than the sum of the basic family of output sets, then the height of the present output sets is upper bounded by the sum of the heights of the basic family of the output sets.

(1c) If

\[ \left( \bigvee_u A^{(i)}(u) \land \bigvee_v B^{(i)}(v) \right) > \bigvee_w C^{(i)}(w), \]

\[ \left( \bigvee_u A'(u) \land \bigvee_v B'(v) \right) \leq \bigvee_i C^{(i)}(w) \] (12a)

and

\[ \left( \bigvee_u A^{(i)}(u) \land \bigvee_v B^{(i)}(v) \right) > \bigvee_z D^{(i)}(z), \]

\[ \left( \bigvee_u A'(u) \land \bigvee_v B'(v) \right) \leq \bigvee_i D^{(i)}(z), \] (12b)
then
\[ C'(w) \subseteq \left( \bigvee_u A'(u) \wedge \bigvee_v B'(v) \right) \quad \text{and} \quad D'(z) \subseteq \left( \bigvee_u A'(u) \wedge \bigvee_v B'(v) \right). \]

That is, if the least height of one of the entries pair of the basic family of input sets is greater than the height of one of the entries of the basic family of the output sets, and the least height of the current input sets is equal to or less than the sum of the heights of the basic family of the output sets, then the height of the present output sets is upper bounded by the least height of the current input sets.

**Example 1.** Let a fuzzy logic controller be described by the following linguistic description:

if \( A(1) \) and \( B(\sim) \) then \( C(1) \)

also

if \( A(2) \) and \( B(2) \) then \( C(2) \)

where

\[
A^{(1)} = [0.7, 0.6, 0.2], \quad B^{(1)} = [0.6, 0.5, 0.4], \quad C^{(1)} = [0.8, 0.6, 0.3],
\]

and

\[
A^{(2)} = [1.0, 0.6, 0.3], \quad B^{(2)} = [0.9, 1.0, 0.8], \quad C^{(2)} = [0.6, 1.0, 0.9].
\]

Let the current input sets be given by

\[
A' = [0.9, 0.6, 0.3] \quad \text{and} \quad B' = [0.8, 0.7, 0.2].
\]

Then

\[
\bigvee_u A^{(1)} = 0.7, \quad \bigvee_v B^{(1)} = 0.6, \quad \bigvee_w C^{(1)} = 0.8;
\]

\[
\left( \bigvee_u A^{(1)} \wedge \bigvee_v B^{(1)} \right) \subseteq \bigvee_w C^{(1)}.
\]

Based upon Proposition 1a, we conclude

\[
C'(w) \subseteq \left( \bigvee_u A' \wedge \bigvee_v B' \right), \quad \bigvee_u A' = 0.9, \quad \bigvee_v B' = 0.8, \quad \text{and} \quad C'(w) \leq 0.8.
\]

(13)

Therefore, we conclude that the current input set \( C'(w) \) is upper bounded by 0.8.

To check if the above conclusion holds true, we compute

\[
C'(w) = A' \circ R_{11} \wedge B' \circ R_{12}
\]

where

\[
R_{11} = \bigvee_{i=1}^{2} \begin{cases} 1, & A^{(i)}(u) \leq C^{(i)}(w), \\ C^{(i)}(w), & A^{(i)}(u) > C^{(i)}(w), \end{cases}
\]

and

\[
R_{12} = \bigvee_{i=1}^{2} \begin{cases} 1, & B^{(i)}(v) \leq C^{(i)}(v), \\ C^{(i)}(v), & B^{(i)}(v) > C^{(i)}(v), \end{cases}
\]

(14a)
Indeed, the conclusion in (13) holds true.

We will investigate the second case now.

Case (ii)

For Case (ii), let us consider the following definitions of the operators:

Aggregation operator: \( \{ \} = \bigcap_i \).

Implication operator: \( \circ \) is Gödel's implication as defined in (7).

Contribution operator: \( \Delta = \land \).

Taking into account Eqs. (4) and the above operators, the present multivariable fuzzy output sets are given by

\[
C'(w) = \bigvee_u \left[ A'(u) \land \bigcap_i^N \left\{ 1, \begin{array}{ll} A^{(i)}(u) = C^{(i)}(w) \\ A^{(i)}(u) > C^{(i)}(w) \end{array} \right. \right] \\
\land \bigvee_v \left[ B'(v) \land \bigcap_i^N \left\{ 1, \begin{array}{ll} B^{(i)}(v) = C^{(i)}(w) \\ B^{(i)}(v) > C^{(i)}(w) \end{array} \right. \right]
\]

Thus, we have

\[
C'(w) = \begin{cases} \\
\bigvee_u A'(u) \land \bigvee_v B'(v), & A^{(i)}(u) \land B^{(i)}(v) \leq C^{(i)}(w), \\
\bigvee_u A'(u) \land \bigvee_v B'(v) \land \bigcap_i^N C^{(i)}(w), & A^{(i)}(u) \land B^{(i)}(v) > C^{(i)}(w), \end{cases}
\]

and,

\[
D'(z) = \bigvee_u \left[ A'(u) \land \bigcap_i^N \left\{ 1, \begin{array}{ll} A^{(i)}(u) = D^{(i)}(z) \\ A^{(i)}(u) > D^{(i)}(z) \end{array} \right. \right] \\
\land \bigvee_v \left[ B'(v) \land \bigcap_i^N \left\{ 1, \begin{array}{ll} B^{(i)}(v) = D^{(i)}(z) \\ B^{(i)}(v) > D^{(i)}(z) \end{array} \right. \right].
\]

Hence it follows that

\[
D'(z) = \begin{cases} \\
\bigvee_u A'(u) \land \bigvee_v B'(v), & A^{(i)}(u) \land B^{(i)}(v) \leq D^{(i)}(z), \\
\bigvee_v A'(u) \land \bigvee_v B'(v) \land \bigcap_i^N D^{(i)}(z), & A^{(i)}(u) \land B^{(i)}(v) > D^{(i)}(z). \end{cases}
\]

In view of Eqs. (15) and (16), we state the following proposition:

**Proposition 2.** (2a) *If*

\[
\left( \bigvee_u A^{(i)}(u) \land \bigvee_v B^{(i)}(v) \right) \leq \bigvee_w C^{(i)} \quad \text{and} \quad \left( \bigvee_u A^{(i)}(u) \land \bigvee_v B^{(i)}(v) \right) \leq \bigvee_z D^{(i)}(z),
\]

\[
R_{11} = \begin{bmatrix} 1.0, & 1.0, & 0.9 \\ 1.0, & 1.0, & 1.0 \\ 1.0, & 1.0, & 1.0 \end{bmatrix}, \quad R_{12} = \begin{bmatrix} 1.0, & 1.0, & 0.9 \\ 1.0, & 1.0, & 1.0 \\ 1.0, & 1.0, & 1.0 \end{bmatrix}, \quad (14b)
\]

\[
C'(w) = [0.9, \ 0.9, \ 0.9] \land [0.8, \ 0.8, \ 0.8] = [0.8, \ 0.8, \ 0.8]. \quad (14c)
\]
then,  
\[ C'(w) \subseteq \left( \bigvee_u A'(u) \wedge \bigvee_v B'(v) \right) \quad \text{and} \quad D'(z) \subseteq \left( \bigvee_u A'(u) \wedge \bigvee_v B'(v) \right). \quad (17) \]

In other words, if the least height of one of the entries pair of the basic family of the input sets is equal to or less than the height of one of the entries of the basic family of the output sets, then the height of the present output sets is upper bounded by the least height of the current input sets.

(2b) If  
\[ \left( \bigvee_u A^{(i)}(u) \wedge \bigvee_v B^{(i)}(v) \right) > \bigvee_w C^{(i)}(w), \]
\[ \left( \bigvee_u A^{(i)}(u) \wedge \bigvee_v B^{(i)}(v) \right) > \bigvee_z D^{(i)}(z), \quad (18a) \]
and  
\[ \left( \bigvee_u A'(u) \wedge \bigvee_v B'(v) \right) \supseteq \bigwedge_i C^{(i)}(w), \]
\[ \left( \bigvee_u A'(u) \wedge \bigvee_v B'(v) \right) \supseteq \bigwedge_i D^{(i)}(z), \quad (18b) \]
then  
\[ C'(w) \subseteq \bigwedge_i C^{(i)}(w) \quad \text{and} \quad D'(z) \subseteq \bigwedge_i D^{(i)}(z). \]

That is, if the least height of one of the entries of the basic family of input sets is greater than the height of one of the entries of the basic family of output sets and the least height of the current input sets is equal to or greater than the intersection of the basic family of output sets, then the height of the present output sets is upper bounded by the intersection of the basic family of the output sets.

(2c) If  
\[ \left( \bigvee_u A^{(i)}(u) \wedge \bigvee_v B^{(i)}(v) \right) > \bigvee_w C^{(i)}(w), \]
\[ \left( \bigvee_u A^{(i)}(u) \wedge \bigvee_v B^{(i)}(v) \right) > \bigvee_z D^{(i)}(z), \quad (19a) \]
and  
\[ \left( \bigvee_u A'(u) \wedge \bigvee_v B'(v) \right) \subseteq \bigwedge_i C^{(i)}(w), \]
\[ \left( \bigvee_u A'(u) \wedge \bigvee_v B'(v) \right) \subseteq \bigwedge_i D^{(i)}(z), \quad (19b) \]
then  
\[ C'(w) \subseteq \left( \bigvee_u A'(u) \wedge \bigvee_v B'(v) \right) \quad \text{and} \quad D'(z) \subseteq \left( \bigvee_u A'(u) \wedge \bigvee_v B'(v) \right). \]

In other words, if the least height of one of the entries of the pair of the basic family of the input sets is greater than the height of one of the entries of the basic family of the output sets and the least height of the current input sets is equal to or less than
the intersection of the basic family of the output sets, then the height of the present output sets is upper bounded by the least height of the current input sets.

**Example 2.** Assume the linguistic description given in Example 1, but using the following set of data:

\[ A^{(1)} = [0.8, 0.9, 0.6], \quad B^{(1)} = [0.7, 0.6, 0.5], \quad C^{(1)} = [0.6, 0.5, 0.4], \]
and
\[ A^{(2)} = [0.6, 0.3, 0.2], \quad B^{(2)} = [0.7, 0.6, 0.4], \quad C^{(2)} = [0.8, 0.9, 1.0]. \]

Let the current input sets be given by

\[ A' = [0.9, 0.2, 0.1], \quad B' = [0.7, 0.6, 0.2]. \]

Then we have

\[ \bigvee_u A^{(1)} = 0.8, \quad \bigvee_v B^{(1)} = 0.7, \quad \bigvee_w C^{(1)} = 0.6; \]

\[ \left( \bigvee_u A^{(1)} \land \bigvee_v B^{(1)} \right) \supset \bigvee_w C^{(1)}, \]
and

\[ \bigvee_u A' = 0.9, \quad \bigvee_v B' = 0.7, \quad \bigvee_i C^{(i)} = C^{(1)} \land C^{(2)} = [0.6, 0.5, 0.4], \]

\[ \left( \bigvee_u A' \land \bigvee_v B' \right) \supset \bigwedge_i C^{(i)}. \]

Based on Proposition 2b, we conclude, therefore, that

\[ C'(w) \subseteq \bigwedge_i C^{(i)}. \]

To check if the above conclusion holds true, we compute

\[ C'(w) = A' \circ R_{11} \land B' \circ R_{12} \]

where

\[ R_{11} = \bigwedge_i \begin{bmatrix} 1, & A^{(i)} \leq C^{(i)}, & 0.6, & 0.5, & 0.4 \\ C^{(i)}, & A^{(i)} > C^{(i)}, & 0.6, & 0.5, & 0.4 \end{bmatrix}, \]

\[ R_{12} = \bigwedge_i \begin{bmatrix} 1, & B^{(i)} \leq C^{(i)}, & 0.6, & 0.5, & 0.4 \\ C^{(i)}, & B^{(i)} > C^{(i)}, & 0.6, & 0.5, & 0.4 \end{bmatrix}, \]

\[ C' = [0.6, 0.5, 0.4] \land [0.6, 0.5, 0.4] = [0.6, 0.5, 0.4]. \]

Indeed, the above conclusion holds true.

**Case (iii)**

For Case (iii), let us consider the following definitions of the operators:

Aggregation operator: \( \oplus' = \bigvee_i \).

Implication operator: \( \odot \) is Godel's implication as defined in (7).
Contribution operator: $\Delta = \nu$.

In this case, the present multivariable fuzzy output sets are expressed as

$$C'(w) = \begin{cases} 
\bigvee_u A'(u) \lor \bigvee_v B'(v), & A^{(i)}(u) \lor B^{(i)}(v) \leq C^{(i)}(w), \\
\bigvee_u A'(u) \lor \bigvee_v B'(v) \land \bigvee_i C^{(i)}(w), & A^{(i)}(u) \lor B^{(i)}(v) > C^{(i)}(w), 
\end{cases}$$

and

$$D'(z) = \begin{cases} 
\bigvee_u A'(u) \lor \bigvee_v B'(v), & A^{(i)}(u) \lor B^{(i)}(v) \leq D^{(i)}(z), \\
\bigvee_u A'(u) \lor \bigvee_v B'(v) \land \bigvee_i D^{(i)}(z), & A^{(i)}(u) \lor B^{(i)}(v) > D^{(i)}(z). 
\end{cases}$$

In view of Eqs. (20) and (21) we state the following proposition:

**Proposition 3.** (3a) If

$$\left( \bigvee_u A^{(i)}(u) \lor \bigvee_v B^{(i)}(v) \right) \leq \bigvee_w C^{(i)}(w),$$

and

$$\left( \bigvee_u A^{(i)}(u) \lor \bigvee_v B^{(i)}(v) \right) \leq \bigvee_z D^{(i)}(z),$$

then

$$C'(w) \subseteq \left( \bigvee_u A'(u) \lor \bigvee_v B'(v) \right) \quad \text{and} \quad D'(z) \subseteq \left( \bigvee_u A'(u) \lor \bigvee_v B'(v) \right).$$

(22)

In other words, if the greatest height of one of the entries pair of the basic family of the input sets is equal to or less than the height of one of the entries of the basic family of the output sets, then the height of the present output sets is upper bounded by the greatest height of the current input sets.

(3b) If

$$\left( \bigvee_u A^{(i)}(u) \lor \bigvee_v B^{(i)}(v) \right) > \bigvee_w C^{(i)}(w),$$

and

$$\left( \bigvee_u A^{(i)}(u) \lor \bigvee_v B^{(i)}(v) \right) > \bigvee_z D^{(i)}(z),$$

then

$$C'(w) \subseteq \bigvee_i C^{(i)}(w) \quad \text{and} \quad D'(z) \subseteq \bigvee_i D^{(i)}(z).$$

(23a) and

$$\left( \bigvee_u A'(u) \lor \bigvee_v B'(v) \right) \geq \bigvee_i C^{(i)}(w),$$

and

$$\left( \bigvee_u A'(u) \lor \bigvee_v B'(v) \right) \geq \bigvee_i D^{(i)}(z),$$

(23b) then

$$C'(w) \subseteq \bigvee_i C^{(i)}(w) \quad \text{and} \quad D'(z) \subseteq \bigvee_i D^{(i)}(z).$$
That is, if the greatest height of one of the entries pair of the basic family of input sets is greater than the height of one of the entries of the basic family of the output sets and the greatest height of the current input sets is equal to or greater than the sum of the basic family of the output sets, then the height of the present output sets is upper bounded by the sum of the heights of the basic family of the output sets.

(3c) If
\[ \left( \bigvee_u A^{(i)}(u) \vee \bigvee_v B^{(i)}(v) \right) > \bigvee_w C^{(i)}(w), \]
\[ \left( \bigvee_u A^{(i)}(u) \vee \bigvee_v B^{(i)}(v) \right) > \bigvee_z D^{(i)}(z), \]
then
\[ C'(w) \subseteq \left( \bigvee_u A'(u) \vee \bigvee_v B'(v) \right) \quad \text{and} \quad D'(z) \subseteq \left( A'(u) \vee \bigvee_v B'(v) \right). \]

That is, if the greatest height of one of the entries pair of the basic family of the input sets is greater than the height of one of the entries of the basic family of the output sets, and the greatest height of the current input sets is equal to or less than the sum of the basic family of the output sets, then the height of the present output sets is upper bounded by the greatest height of the current input sets.

Example 3. Let the following basic families of input–output sets be given:
\[ A^{(1)} = [0.9, 0.7, 0.5], \quad B^{(1)} = [0.7, 0.6, 0.4], \quad C^{(1)} = [0.8, 0.6, 0.8], \]
and
\[ A^{(2)} = [1.0, 0.9, 0.2], \quad B^{(2)} = [0.4, 0.6, 0.8], \quad C^{(2)} = [0.9, 0.8, 0.6]. \]

We assume the current input sets as follows:
\[ A' = [0.8, 0.5, 0.2] \quad \text{and} \quad B' = [0.6, 0.4, 0.2]. \]

Then we have
\[ \bigvee_u A^{(1)} = 0.9, \quad \bigvee_v B^{(1)} = 0.7, \quad \bigvee_w c^{(1)} = 0.8; \quad \left( \bigvee_u A^{(1)} \vee \bigvee_v B^{(1)} \right) > \bigvee_w C^{(1)}, \]
and
\[ \bigvee_u A' = 0.8, \quad \bigvee_v B' = 0.6, \]
\[ \bigvee_i C^{(i)} = [0.9, 0.8, 0.8], \quad \left( \bigvee_u A' \vee \bigvee_v B' \right) > \bigvee_i C^{(i)}. \]

Based on Proposition 3c, we infer, therefore, that
\[ C'(w) \subseteq \left( \bigvee_u A' \vee \bigvee_v B' \right). \]

The present output is upper bound by 0.8.
Let us now compute

\[ C'(w) = A' \circ R_{11} \lor B' \circ R_{12} \]

where

\[
R_{11} = \begin{bmatrix} 0.9, & 0.8, & 0.8 \\ 1.0, & 0.8, & 1.0 \\ 1.0, & 1.0, & 1.0 \end{bmatrix}, \quad R_{12} = \begin{bmatrix} 1.0, & 1.0, & 1.0 \\ 1.0, & 1.0, & 0.6 \end{bmatrix},
\]

\[
C' = [0.8, 0.8, 0.8] \lor [0.6, 0.6, 0.6] = [0.8, 0.8, 0.8].
\]

Indeed, the above conclusion holds true.

**Case (iv)**

For Case (iv), let us consider the following definitions of the operators:

- Aggregation operator \( \oplus = \lor \).
- Implication operator: \( \ominus \) is Gödel’s implication as defined in (7).
- Contribution operator: \( \Delta = \lor \).

Using (4), the present multivariable fuzzy output sets are stated as

\[
C'(w) = \begin{cases} 
\lor_u A'(u) \lor \lor_v B'(v), & A^{(0)}(u) \lor B^{(0)}(v) \leq C^{(0)}(w), \\
\lor_u A'(u) \lor \lor_v B'(u) \land \lor_i C^{(0)}(w), & A^{(0)}(u) \lor B^{(0)}(v) > C^{(0)}(w), 
\end{cases}
\]

\[ (25) \]

\[
D'(z) = \begin{cases} 
\lor_u A'(u) \lor \lor_v B'(v), & A^{(0)}(u) \lor B^{(0)}(v) \leq D^{(0)}(z), \\
\lor_u A'(u) \lor \lor_v B'(u) \land \lor_i D^{(0)}(z), & A^{(0)}(u) \lor B^{(0)}(v) > D^{(0)}(z), 
\end{cases}
\]

\[ (26) \]

Considering Eqs. (25) and (26), we state the following proposition:

**Proposition 4.** (4a) If

\[
\left( \lor_u A^{(0)}(u) \lor \lor_v B^{(0)}(v) \right) \leq \lor_w C^{(0)}(w)
\]

and

\[
\left( \lor_u A^{(0)}(u) \lor \lor_v B^{(0)}(v) \right) \leq \lor_z D^{(0)}(z),
\]

then

\[
C'(w) \leq \left( \lor_u A'(u) \lor \lor_v B'(v) \right) \quad \text{and} \quad D'(z) \leq \left( \lor_u A'(u) \lor \lor_v B'(v) \right). \quad (27)
\]

In other words, if the greatest height of one of the entries pair of the basic family of the input sets is equal to or less than the height of one of the entries of the basic family of the output sets, then the height of the present output sets is upper bounded by the greatest height of the current input sets.

(4b) If

\[
\left( \lor_u A^{(0)}(u) \lor \lor_v B^{(0)}(v) \right) > \lor_w C^{(0)}(w),
\]

\[
\left( \lor_u A^{(0)}(u) \lor \lor_v B^{(0)}(v) \right) > \lor_z D^{(0)}(z),
\]

\[ (28a) \]
and

\[
\bigg( \bigvee_{u} A'(u) \lor \bigvee_{v} B'(v) \bigg) \supseteq \bigwedge_{i} C^{(i)}(w),
\]

\[
\bigg( \bigvee_{u} A'(u) \lor \bigvee_{v} B'(v) \bigg) \supseteq \bigwedge_{i} D^{(i)}(z),
\]

then

\[
C'(w) \subseteq \bigwedge_{i} C^{(i)}(w) \quad \text{and} \quad D'(z) \subseteq \bigwedge_{i} D^{(i)}(z).
\]

That is, if the greatest height of one of the entries pair of the basic family of the input sets is greater than the height of one of the entries of the basic family of output sets and the greatest height of the current input sets is equal to or greater than the intersection of the basic family of output sets, then the height of the present output sets is upper bounded by the intersection of the basic family of the output sets.

(4c) If

\[
\bigg( \bigvee_{u} A^{(i)}(u) \lor \bigvee_{v} B^{(i)}(v) \bigg) > \bigvee_{w} C^{(i)}(w),
\]

\[
\bigg( \bigvee_{u} A^{(i)}(u) \lor \bigvee_{v} B^{(i)}(v) \bigg) > \bigvee_{z} D^{(i)}(z),
\]

and

\[
\bigg( \bigvee_{u} A'(u) \lor \bigvee_{v} B'(v) \bigg) \subseteq \bigwedge_{i} C^{(i)}(w),
\]

\[
\bigg( \bigvee_{u} A'(u) \lor \bigvee_{v} B'(v) \bigg) \subseteq \bigwedge_{i} D^{(i)}(z),
\]

then

\[
C'(w) \subseteq \bigg( \bigvee_{u} A'(u) \lor \bigvee_{v} B'(v) \bigg) \quad \text{and} \quad D'(z) \subseteq \bigg( \bigvee_{u} A'(u) \lor \bigvee_{v} B'(v) \bigg).
\]

In other words, if the greatest height of one of the entries pair of the basic family of the input sets is greater than the height of one of the basic family of output sets and the greatest height of the current input sets is equal to or less than the intersection of the basic family of the output sets, then the height of the present output sets is upper bounded by the greatest height of the current input sets.

Example 4. Using the linguistic description in Example 1, the basic families of input–output sets, and the current input sets given in Example 2, we have

\[
\bigvee_{u} A^{(2)} = 0.6, \quad \bigvee_{v} B^{(2)} = 0.7, \quad \bigvee_{w} C^{(2)} = 0.8;
\]

\[
\bigg( \bigvee_{u} A^{(2)} \lor \bigvee_{v} B^{(2)} \bigg) < \bigvee_{w} C^{(2)}.
\]

Based upon Proposition 4a, we infer,

\[
C'(w) \subseteq \bigg( \bigvee_{u} A'(u) \lor \bigvee_{v} B'(v) \bigg),
\]
### Table 1. Conditions for upper bounds of Gödel multivariable fuzzy controller

<table>
<thead>
<tr>
<th>Upper Bounds</th>
<th>$\alpha^i_1 = \vee_i$</th>
<th>$\alpha^i_2 = \vee_i$</th>
<th>$\alpha^i_3 = \wedge_i$</th>
<th>$\alpha^i_4 = \wedge_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\vee_i A^{(i)} \wedge \vee_v B^{(i)} \geq \vee_w C^{(i)}$, $\vee_u A' \wedge \vee_v B' \geq \vee_i C^{(i)}$</td>
<td>$1) \vee_u A^{(i)} \wedge \vee_v B^{(i)} &gt; \vee_w C^{(i)}$, $\vee_u A' \wedge \vee_v B' \geq \vee_i C^{(i)}$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td>$\wedge_i C^{(i)}$</td>
<td>$-$</td>
<td>$-$</td>
<td>$1) \vee_u A^{(i)} \wedge \vee_v B^{(i)} &gt; \vee_w C^{(i)}$, $\vee_u A' \wedge \vee_v B' \geq \wedge_i C^{(i)}$</td>
<td>$-$</td>
</tr>
<tr>
<td>$\vee_u A' \vee \vee_v B'$</td>
<td>$-$</td>
<td>$-$</td>
<td>$1) \vee_u A^{(i)} \wedge \vee_v B^{(i)} &gt; \vee_w C^{(i)}$, $\vee_u A' \wedge \vee_v B' \geq \wedge_i C^{(i)}$</td>
<td>$1) \vee_u A^{(i)} \wedge \vee_v B^{(i)} &gt; \vee_w C^{(i)}$, $\vee_u A' \wedge \vee_v B' \geq \wedge_i C^{(i)}$</td>
</tr>
<tr>
<td>$\vee_u A' \wedge \vee_v B'$</td>
<td>$1) \vee_u A^{(i)} \wedge \vee_v B^{(i)} \leq \vee_w C^{(i)}$, $\vee_u A' \wedge \vee_v B' \leq \vee_i C^{(i)}$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td>$\vee_u A' \wedge \vee_v B'$</td>
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<td>$\vee_u A' \wedge \vee_v B'$</td>
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<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
</tbody>
</table>
where
\[ \bigvee_u A' = 0.9, \quad \bigvee_v B' = 0.9, \quad \text{i.e.} \quad C'(w) \leq 0.9. \]

So the present output set \( C'(w) \) is upper bounded by 0.9.

To check the above results, we compute
\[
C'(w) = A' \circ R_{11} \lor B' \circ R_{12}
\]
where \( R_{11} \) and \( R_{12} \) are given in Example 2, and find \( C' = [0.6, 0.5, 0.4] \). The conclusion is therefore true.

Remarks. The results obtain in this section are summarized in Table 1. Some general conclusions can be drawn based on Propositions 1 to 4 and Table 1.

There are two categories of upper bounds:
(i) upper bounds generated by the basic family of output sets shown in Figure 4 and
(ii) upper bounds created by the current input sets as depicted in Figure 5.

These categories may have optimistic bounds produced by the max-operator, that is
\[
\bigvee_i C^{(i)}(w), \quad \left[ \bigvee_u A' \lor \bigvee_v B' \right],
\]
and pessimistic bounds made by the min-operator,
\[
\bigwedge_i C^{(i)}(w), \quad \left[ \bigvee_u A' \lor \bigvee_v B' \right].
\]

The construction of the basic families of the input–output sets and the current input sets determine an appropriate upper bound for the present output sets.

The optimistic upper bound \( \bigvee_i^{N} C^{(i)}(w) \) is created for the aggregation operator

![Fig. 4. Basic family of output bounds.](image-url)
Multivariable fuzzy controller

\[ A_{(u)} \]

\[ B_{(v)} \]

\[ C_{(w)} \]

Fig. 5. Current input bounds.

\[ \mathcal{A}' = \bigvee_i, \text{ and for the contribution operator } \Delta = \land, \text{ or } \Delta = \lor. \text{ The least height of the current inputs sets should be equal to or greater than the sum of the heights of the basic family of the outputs sets and the least height of one of an entry pair of the basic family of the input sets should be greater than the height of one of the sets of the basic family of the output sets to reach } \bigvee_i^N C^{(i)}(w). \text{ Therefore, choosing contribution operator } \Delta \text{ as } \max (\lor) \text{ seems to be more suitable in this case.} \]

The optimistic upper bound \( \{ \bigvee_u A' \lor \bigvee_v B' \} \) is generated for the contribution operator \( \Delta = \lor \), and the aggregation operator \( \mathcal{A}' = \bigvee_i \), or \( \mathcal{A}' = \bigwedge_i \). However, the upper bound conditions for \( \mathcal{A}' = \bigvee_i \) and \( \Delta = \lor \) are ‘weaker’ than for \( \mathcal{A}' = \bigvee_i \) and \( \Delta = \lor \). Therefore, we recommend \( \mathcal{A}' = \bigvee_i \) and \( \Delta = \lor \) as a combination of aggregation and contribution operators to create the greatest upper bound \( \{ \bigvee_i^N C^{(i)}(w) \lor [\bigvee_u A' \lor \bigvee_v B'] \} \). In general \( [\bigvee_u A' \lor \bigvee_v B'] \equiv C^{(i)}(w) \). We can see, therefore, that the greatest upper bound \( [\bigvee_u A' \lor \bigvee_v B'] \) is generated when \( \mathcal{A}' = \bigvee_i \), \( \Delta = \lor \) on the condition that

\[ \bigvee_u A' \lor \bigvee_v B' \leq \bigvee_w C^{(i)} \]  \hspace{1cm} (30)

and

\[ \bigvee_u A^{(i)} \lor \bigvee_v B^{(i)} > \bigvee_w C^{(i)}, \quad \bigvee_u A' \lor \bigvee_v B' \leq C^{(i)}. \]  \hspace{1cm} (31)

The pessimistic upper bound \( \bigwedge_i^N C^{(i)} \) is generated when \( \mathcal{A}' = \bigwedge_i \) and \( \Delta = \land \), or \( \Delta = \lor \). In this case choosing contribution operator as \( \max (\lor) \) requires ‘weaker’
upper bound conditions than as min (A). The pessimistic upper bound \( \bigvee \bar{u} \bar{A} \wedge \bigvee \bar{v} \bar{B} \) is created when \( \Delta = \wedge, \bar{\alpha} = \bigvee \) or \( \bar{\alpha} = \bigwedge \).

However, the upper bound conditions for \( \bar{\alpha} = \bigvee \) are more ‘elastic’ than those for \( \bar{\alpha} = \bigwedge \). Usually, \( \bigvee \bar{u} \bar{A} \wedge \bigvee \bar{v} \bar{B} \) \( \geq \bigwedge \bar{u} \bar{C}^{(i)}(w) \). We can therefore conclude that the least upper bound \( \bigwedge \bar{u} \bar{C}^{(i)} \) is produced when \( \bar{\alpha} = \bigwedge, \Delta = \vee \) on the condition that

\[
\left[ \bigvee \bar{u} \bar{A}^{(i)} \vee \bigvee \bar{v} \bar{B}^{(i)} \right] > C^{(i)} \tag{32}
\]

and

\[
\left[ \bigvee \bar{u} \bar{A} \vee \bigvee \bar{v} \bar{B} \right] \equiv \bigwedge \bar{u} \bar{C}^{(i)} \tag{33}
\]

We see that the greatest upper bound \( \bigvee \bar{u} \bar{A} \vee \bigvee \bar{v} \bar{B} \) as well as the least upper bound \( \bigwedge \bar{u} \bar{C}^{(i)} \) are generated when the contribution operator \( \Delta = \max (\vee) \). Finally, it is proper to choose the contribution operator \( \Delta = \max (\vee) \) when one uses the aggregation operator \( \bar{\alpha} = \max (\vee) \) or \( \bar{\alpha} = \min (\wedge) \) and choosing fuzzy implication as Gödel’s implication in the design of the multivariable fuzzy controller.

4. Conclusions

In this paper, we have analyzed the upper bounds for multivariable fuzzy controller outputs. The generalized multivariable structure of the fuzzy controller has been proposed. This structure consists of composition operators, contribution operators, aggregation operators and implication operators. Such construction incorporates a translating stage and an inferring stage and combines the fuzzy rules with the inference mechanism into a unified form of the multivariable fuzzy equations. Gödel’s fuzzy implication has been used to investigate the multivariable upper bounds for the present output fuzzy sets.

Two categories of upper bounds have been found. The construction of the basic families of the input–output fuzzy sets and the structure of the current input sets determine an appropriate upper bound for the present output fuzzy sets.

The greatest upper bound \( \bigvee \bar{u} \bar{A} \vee \bigvee \bar{v} \bar{B} \) is generated when \( \bar{\alpha} = \bigvee, \Delta = \vee \). The least upper bound \( \bigwedge \bar{u} \bar{C}^{(i)} \) is created when \( \bar{\alpha} = \bigwedge, \Delta = \vee \). Choosing contribution operator \( \Delta \) as \( \max (\vee) \) ensures the greatest upper bound as well as the least upper bound.

The general conclusions drawn from these investigations are that the designed system is able to evaluate the values of the upper bounds of the present outputs of the multivariable fuzzy controller theoretically, hence without the need of expensive simulations. Thus, the designer has freedom in choosing appropriate fuzzy operators and constructing the basic family of input–output sets and the structure of current fuzzy input sets to obtain desirable features, of the multivariable present output fuzzy sets.
Multivariable fuzzy controller

References