Decoupling of descriptor systems

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Abstract: Using the descriptor standard form, the decoupling problem of the descriptor system by state feedback is studied. The descriptor system is first transformed in controller form and the necessary and sufficient conditions for the descriptor system to be integrator decoupled are obtained. Then, these conditions are shown to be necessary and sufficient for the closed-loop system poles to be assigned while decoupling. The characteristic polynomial of the closed-loop system is explicitly calculated and the conditions for the decoupled system to be internally stable are obtained.

1 Introduction

System decoupling or noninteracting control is one of the major problems in multivariable system theory and has been extensively discussed in a number of papers on linear regular systems [1–5]. Some literature has also appeared on the decoupling of the descriptor systems [6–9]. In particular, Christodoulou [7] considered the decoupling problem for the descriptor systems using proportional and derivative feedback where Pandolfi’s transformation and the duality played a key role in deriving the similar necessary and sufficient conditions that have been established for decoupling in the regular systems. Zhou et al. [8] also gave similar necessary and sufficient conditions for the decoupling of the descriptor systems using modified proportional and derivative feedback. Dai [9] studied the static decoupling as well as the dynamic decoupling using proportional and derivative feedback. He showed that the closed-loop transfer matrix can be diagonalised with the specified polynomials at its diagonal entries if and only if the open-loop transfer matrix is nonsingular.

In this paper, the decoupling of the descriptor system is considered using only the proportional feedback of the state so that the closed-loop transfer matrix may be diagonalised with the proper rational functions at its diagonal entries. This problem is more important than the one considered using proportional and derivative feedback and cannot be reduced to that of the regular system by using simple transformations.

2 Problem statement

Consider the descriptor system defined by

\[ E_1 \dot{x} = A_1 x + B_1 u \]  \hspace{1cm} \[ y = C x \]

where \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^m \) and \( y \in \mathbb{R}^p \) are the state vector, the input vector and the output vector, respectively, and \( \{A_1, B_1, C, E_1\} \) are constant matrices of appropriate dimensions. We assume that

(1) The system (eqn. 1a) is regular.

(2) The transfer matrix \( H(s) \) has a maximum rank, i.e.

\[ \text{rank} \{H(s)\} = \text{rank} \{C(sE_1 - A_1)^{-1}B_1\} = m \]

(3) The system (eqn. 1a) is stabilisable.

From assumption 1, there exists a real number \( \sigma \) such that \( \sigma E_1 - A_1 \neq 0 \). Select such a \( \sigma \in \mathbb{R} \), add \( -\sigma E_1 x \) to both sides of eqn. 1a and multiply the resulting equation by \( (\sigma E_1 - A_1)^{-1} \) from the left. Then, the system (eqn. 1) can equivalently be expressed in the following descriptor standard form:

\[ E(\dot{x} - \sigma x) = x + Bu \] \hspace{1cm} \[ y = C x \]

where \( E \triangleq (\sigma E_1 - A_1)^{-1} E_1 \) and \( B \triangleq (\sigma E_1 - A_1)^{-1} B_1 \). In the sequel, the system (eqn. 3) is considered as the starting equation and is designated by the triplet \((C, E, B)\).

Now, observe that if the following state feedback

\[ u = -Fx + Gv \]

is employed in eqn. 3, the resulting closed-loop system becomes

\[ E(\dot{x} - \sigma x) = (I - BF)x + BGv \] \hspace{1cm} \[ y = C x \]

Therefore, the transfer matrix \( H(F, G, s) \) is calculated as

\[ H(F, G, s) = C(\sigma E - I + BF)^{-1}BG \]

with \( \tilde{s} = s - \sigma \). For the closed-loop system (eqn. 5) to have a unique solution the following condition is imposed on the feedback gain matrix \( F \):

\[ |\sigma E - I + BF| \neq 0 \]

The problem of the decoupling is to find the necessary and sufficient conditions for the existence of an appropriate decoupling pair \((F, G)\) so that the closed-loop transfer matrix \( H(F, G, s) \) may be diagonal and nonsingular.
In eqn. 3, we may assume, without loss of generality, that the pair \((E, B)\) is controllable in a usual sense, i.e.
\[
\text{rank } [B \ E 
\begin{bmatrix} E_{11} & \cdots & E_{1r} \\ \vdots & \ddots & \vdots \\ E_{r1} & \cdots & E_{rr} \end{bmatrix}] = n
\] (8)
Otherwise, there exists an appropriate co-ordinate transformation \(T\) such that the new system matrices become
\[
E = TET^{-1} = \begin{bmatrix} E_{11} & \cdots & E_{1r} \\ \vdots & \ddots & \vdots \\ E_{r1} & \cdots & E_{rr} \end{bmatrix},
\]
\[
B = TB = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}
\]
\[
\bar{C} = CT^{-1} = \begin{bmatrix} C_1 & \cdots & C_r \\ \vdots & \ddots & \vdots \\ C_r & \cdots & C_1 \end{bmatrix}
\]
\[
F = FT^{-1} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}
\]
(9)
with the pair \((\bar{E}_1, \bar{B}_1)\) controllable, whereas the transfer matrix
\[
H(F, G, S) = C\bar{E}_1 + B_1G \] (10)
remains the same and the suppressed uncontrollable modes determined by \(|\bar{E}_{22} - I| = 0\) are strictly stable by assumption 2.

3 Integrator decoupling problem

In this Section, the decoupling problem is considered of finding the conditions such that the transfer matrix \(H(F, G, S)\) is integrator decoupled, i.e.
\[
H(F, G, S) = \text{diag } [C, \cdots, C,] \]
\[
F = m^{-1} = [F, \cdots, F] \] (9)
\[
B_1 = \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & b_{m-1m} \end{bmatrix}
\]
(11)
and \(B_1, \bar{C}, E_0\) and \(F_0\) are defined, respectively, as
\[
\bar{C} = [C_{ij}]; C_{ij} = [c_{ij, 1}, c_{ij, 2}, \ldots, c_{ij, m}] \in R^{n \times m},
\]
\[
E_0 = [E_{ij}]; E_{ij} = [e_{ij, 1}, e_{ij, 2}, \ldots, e_{ij, m}] \in R^{n \times m},
\]
\[
F_0 = [F_{ij}]; F_{ij} = [f_{ij, 1}, f_{ij, 2}, \ldots, f_{ij, m}] \in R^{n \times m}.
\]

Note that \(\bar{C}\) is blockwise column partitioned so that \(C_{ij}\) is a \(\mu_j\) dimensional row vector. \(E_0\) and \(F_0\) are partitioned in a similar way. Using these notations, \(H(F, G, S)\) can alternately be expressed as follows:

Lemma 1
\[
H(F, G, S) = (-S)\bar{C}(S - B_0)_{-1}^i \times B_0[I - F_0B_0 - (E_0 + F_0D_0)_{-1}^iG
\]
\[
+ (S - D_0)_{-1}^iB_0G
\]
\[
\text{where } \bar{S} = (S - \sigma)^{-1}.
\]

Proof
Observe that from eqn. \(13\)
\[
H(F, G, S) = \begin{bmatrix} \bar{C}(S - B_0)_{-1}^i & \bar{C}(S - B_0)_{-1}^i & \cdots & \bar{C}(S - B_0)_{-1}^i \end{bmatrix}
\]
\[
\times \begin{bmatrix} I - F_0B_0 - (E_0 + F_0D_0)_{-1}^iG \\ \vdots \\ I - F_0B_0 - (E_0 + F_0D_0)_{-1}^iG \end{bmatrix}
\]
\[
\times (S - D_0)_{-1}^iB_0G
\]
where \(\bar{S} = (S - \sigma)^{-1}.
\]

Applying the matrix inversion lemma to the term in square brackets in eqn. 17 yields
\[
[I - (E_0 + F_0D_0)(S - D_0)_{-1}^iB_0G_{-1}]^{-1}
\]
\[
= I - (E_0 + F_0D_0)(S - D_0)_{-1}^iB_0G_{-1} \]
\[
[\text{above}] = I - (E_0 + F_0D_0)(S - D_0)_{-1}^iB_0G_{-1} \] (18)

\[102x748\]
Substituting eqn. 18 into eqn. 17 now yields

\[ H(F, G, S) = (-\bar{S}) (S1 - D_0)^{-1} \times B_0 [I - (E_0 + 3F_0)(S1 - D_0)^{-1}B_0]^{-1} \bar{G} \]  

(19)

and hence the lemma follows.

Based on lemma 1, the necessary and sufficient conditions for the system (eqn. 3) to be integrator decoupled will now be derived. For this, suppose that the system is integrator decoupled as in eqn. 1. Then, substituting eqn. 1 into eqn. 16 yields

\[ \bar{S} \cdot \text{diag} [3^{-\sigma_1} \ldots 3^{-\sigma_n}] \bar{C} (S1 - D_0)^{-1} B_0 \]

\[ = \bar{G} \cdot [I - F_0 B_0 - (E_0 + F_0 D_0) (S1 - D_0)^{-1} B_0] \]  

(20)

Because in eqn. 20 the right hand side is a polynomial matrix in \( \bar{S} = \sigma - \bar{S}^{-1} \), so is the left hand side, which can be rewritten as

\[ \bar{S} \cdot \text{diag} [3^{-\sigma_1} \ldots 3^{-\sigma_n}] \bar{C} (S1 - D_0)^{-1} B_0 \]

\[ = B^* + C^* (S1 - D_0)^{-1} B_0 + D^*(\bar{S}) \times \text{diag} [3^{-\sigma_1} \ldots 3^{-\sigma_n}] \]  

(21)

where

\[ B^* = [b^*_i] \]

\[ C^* = [C^*_i] \]

\[ D^*(\bar{S}) = [d^*_i(\bar{S})] \]

with the elements \((i, j, k = 1, 2, \ldots, m)\) given by

\[ b^*_i = c_{ij} - \sigma_i + 1 \]  

(23a)

\[ C^*_i = [c_{ij, 1 - \sigma_i + 1}, c_{ij, 2 - \sigma_i + 1}, \ldots, c_{ij, m - \sigma_i + 1}] \]  

(23b)

\[ d^*_i(\bar{S}) = \begin{cases} 0 & (\sigma_i < 2) \\ \sum_{a_1 + \ldots + a_m = \sigma_i - 1} c_{ij, a_1} \ldots c_{ij, a_m} 3^{a_1 - 1} + \ldots + c_{ij, a_1} \ldots 3^{(\sigma_i - 2)} & (\sigma_i \geq 2) \end{cases} \]  

(23c)

In the above, the following convention is used.

\[ c_{ij, k} = 0 \]

for \( k = 0 \) or \( \mu_j + 1 \leq k \)

Because \((D_0, B_0)\) is a controllable pair, comparing the powers of \( \bar{S}^{-1} \) in eqn. 21 with those of the right hand side in eqn. 20 gives

\[ B^* = -\bar{G}^{-1}(I - F_0 B_0) \]

(24a)

\[ C^* = \bar{G}^{-1}(E_0 + F_0 D_0) \]

(24b)

\[ D^*(\bar{S}) = 0 \]  

(24c)

Consequently, \((C, E, B)\) is integrator decoupled with the integrator decoupling indices \( \{\sigma_1, \ldots, \sigma_n\}\) only if

(a) a decoupling pair \((F_0, \bar{G})\) satisfying eqns. 24a and 24b.

(b) \( c_{ij, k} = c_{ij, k - 1} + c_{ij, k - 3} + \ldots + c_{ij, 0} = 0 \),

for \( \sigma_i \geq 2 \) \((i, j = 1, 2, \ldots, m)\) (25)

As for the solvability of eqns. 24a and 24b with respect to the free variables \( F_0 \) and \( \bar{G} \), note that

\[ F_0 B_0 = \begin{bmatrix} f_{11, 1} f_{12, 1} & \cdots & f_{1m, 1} \\ \vdots & \ddots & \vdots \\ f_{m1, 1} & \cdots & f_{mm, 1} \end{bmatrix} \]

\[ F_0 D_0 = \begin{bmatrix} f_{11, 1} & \cdots & f_{11, m} & 0 & \cdots & f_{1m, m} & 0 \\ \vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \vdots \\ f_{m1, 1} & \cdots & f_{m1, m} & 0 & \cdots & f_{mm, m} & 0 \end{bmatrix} \]

Hence, given any \( B^* \) and \( \bar{G} \), eqn. 24a is always solvable by the proper choice of \( \{F_{01}, \ldots, F_{0m}\} \). Similarly, given any \( C^* \), \( E_0 \) and \( \bar{G} \), eqn. 24b is solvable for \( F_0 \) except the \( \rho_1 = \mu_1 \)th, the \( \rho_2 = \mu_2 + \mu_3 \)th, \ldots and the \( \rho_n = \mu_1 + \mu_2 + \mu_3 + \ldots + \mu_n \)th columns. Let \( \Gamma \) denote the operator that extracts these \( \rho_1 \)th, \( \rho_2 \)th, \ldots and \( \rho_n \)th columns out of \( n \) columns and let

\[ C^* \triangleq \Gamma C^* \]

\[ E_0 \triangleq \Gamma E_0 \]

\[ F_0 = (\bar{G} B^* B_0 + B_0 + \bar{G} C^* D_0 - E_0 D_0) = B_{\Gamma}^{-1} F_0 T \]

(33)
where $J$ is an arbitrary matrix that makes $| \tilde{G} | \neq 0$. $P$ is a permutation matrix defined by $P = P(i_0, i_1) \cdots P(1, i_1)$ with $P(k, i_k)$ a permutation matrix that interchanges the $k$th and $i_k$th rows, and the matrices $B^*$ and $C^*$ are calculated by eqn. 23 using the decoupling indices (eqn. 31).

Proof

The 'only if' part of the theorem is already established. To show the 'if' part, note first that the way $r_1$ and $\sigma_i$ are defined, eqn. 25 is automatically satisfied, and that $C^*$ in eqn. 26 consists of the zero rows except the $(i_1, i_2, \ldots, i_q)$th rows, i.e.

$$PC^* = \begin{bmatrix} C_q \\ 0 \end{bmatrix}$$

Therefore, using eqn. 32, $\tilde{G}C^*$ is seen to satisfy

$$\tilde{G}C^* = [E_p C_4(C_0)^{-1} J] C_q 0$$

$$= E_p C_4(C_4)^{-1} C_q$$

$$= E_p$$

(35)

On the other hand, note that eqns. 24a and 24b can be rewritten simultaneously as

$$F_0[B_0 D_0] = [GB^* + I \tilde{G}C^* - E_0]$$

(36)

Because by eqn. 25 there exists a solution $F_0$ satisfying eqn. 36, such a unique $F_0$ is given by

$$F_0 = (GB^* + I)B_0 + (\tilde{G}C^* - E_0)D_0$$

(37)

This completes the proof.

Note that as the theorem indicates, there may be multiple sets of the integrator decoupling indices $[\sigma_1, \ldots, \sigma_m]$ for which eqns. 25 and 27 are satisfied.

4 Pole assignment

The problem of finding the necessary and sufficient conditions for the existence of the decoupling pair $(F, G)$, such that the poles of the closed-loop system are assigned in the specified location while decoupling eqn. 3, will now be considered.

Suppose that the system (eqn. 11), described in controller form, is decoupled as

$$H_i(F, G, S) = \frac{1}{S^{n_1} + a_{11}S^{n_1+1} + \cdots + a_{1m}S^m}$$

(38)

where $H_i(F, G, S)$ is the $i$th row of $H(F, G, S)$ and $e_i$ is the $1 \times m$ unit row vector with zeros but a one in the $i$th position and the non-negative integers $[\sigma_1, \ldots, \sigma_m]$ are referred to as the decoupling indices. Then, using lemma 1 yields

$$(-S)^{\delta} \text{diag} [h_1(S), \ldots, h_m(S)]  \tilde{C}(I - D_0)^{-1} B_0$$

$$= \tilde{G}^{-1}[I - F_0 B_0 - (E_0 + F_0 D_0)S(I - D_0)^{-1} B_0]$$

(39)

Here, the left hand side of eqn. 39 is rewritten as

$$\delta \text{diag} [h_1(S), \ldots, h_m(S)] \tilde{C}(I - D_0)^{-1} B_0$$

$$= B^{**} + C^{**}(I - D_0)^{-1} B_0 + D^{**}(S) \times \text{diag} [S^{-(a_1+1)} \cdots S^{-(a_m+1)}]$$

(40)

where

$$B^{**} = [h_0^{**}]$$

$$C^{**} = [C_0^{**}]$$

$$D^{**}(S) = [d_0^{**}(S)]$$

and

$$h_0^{**} = \begin{bmatrix} \sigma_1 < 2 \\ k(3) \end{bmatrix}$$

(41a)

$$C_0^{**} = \begin{bmatrix} \sum \sigma_i - \sigma_j \sigma_i \end{bmatrix}$$

(41b)

with

$$k(3) = c_{ij} + c_{ij} - c_{ij}$$

(41c)

In the above, the following convention is used:

$$c_{ij} = 0 \quad \text{for } k \leq 0 \quad \text{or } \mu_j + 1 \leq k$$

$$c_{ij} = 1 \quad \text{for } k = 0$$

(42a)

Substituting eqn. 40 into eqn. 39 and comparing the powers of $S^{-1}$ in eqn. 39 yields

$$B^{**} = -\tilde{G}^{-1}(I - F_0 B_0)$$

(42b)

$$C^{**} = \tilde{G}^{-1}(E_0 + F_0 D_0)$$

(42c)

Note that eqn. 42 is valid if and only if

$$c_{ij} = c_{ij}, \quad \cdots = c_{ij, \sigma_j + 2} = 0$$

(43)

The solvabilities of eqns. 42a and 42b are similar to those of eqns. 24a and 24b. In particular, let $C_{**} = \Gamma C^{**}$ be a matrix comprising the $(p_1, p_2, \ldots, p_m)$ columns of $C^{**}$. Then the following important identity can be checked easily:

$$C_{p} = C^*$$

Thus, the following theorem is established.

Theorem 2

The closed-loop system poles can be arbitrarily assigned while simultaneously decoupling eqn. 3 with the decoupling indices $[\sigma_1, \ldots, \sigma_m]$ if and only if $(C, E, B)$ can be integrator decoupled with the same decoupling indices $[\sigma_1, \ldots, \sigma_m]$. An appropriate set of decoupling indices and a decoupling pair $(F, G)$ are given, respectively, by eqn. 31 and

$$G = B_1^{-1} \tilde{G}$$

(44a)

$$F = B_1^{-1}(\tilde{G}B^*B_0 + B_0 + \tilde{G}C^*D_0 - E_0 D_0)T$$

(44b)

where $G$ is the same matrix defined earlier.
The following theorem shows where the poles of the closed-loop system will be assigned when the system is decoupled.

**Theorem 3**

The characteristic polynomial of the closed-loop system using the control law eqn. 4 which decouples \((C, E, B)\) as eqn. 38 is given by

\[
\| \mathbf{z}E - I + BF \| = (-1)^{r+m} | \mathbf{C'} | \mathbf{C'(s)}
\]

\[
\times \{ \prod_{i=1}^{m} h_i(s) \} \neq 0
\]

(45)

where \( \mathbf{C'(s)} = \mathbf{C'(s)} \mathbf{I} \mathbf{D}^{-1} \mathbf{B} \) is an \( m \times m \) polynomial matrix satisfying \( | \mathbf{C'(s)} | \neq 0 \). Here, the \( (i, j) \)th element of \( \mathbf{C'(s)} \) is given by

\[
\tilde{c}_{ij}(s) = c_{ij} + c_{ij} \bar{s}^{-1} + \cdots + c_{ij} \nu \bar{s}^{-\nu+1}
\]

\[(i, j = 1, 2, \ldots, m) \quad (46)\]

Proof

Observe that by eqns. 14 and 39

\[
| \mathbf{z}E - I + BF | = (-1)^{r+m} | \mathbf{C'} | \mathbf{C'(s)}
\]

\[
\times \{ \prod_{i=1}^{m} h_i(s) \} \neq 0
\]

(45)

which proves eqn. 45. To show \( | \mathbf{C'(s)} | \neq 0 \), suppose contrarily that \( | \mathbf{C'(s)} | = 0 \). Then, some row of \( \{ \tilde{c}_{ij}(s) \} \), say, the \( k \)th is a linear combination of the other rows, i.e. \( \{ \tilde{c}_{ij}(s) \} \mathbf{k} = 0 \). This, however, implies that by the same elementary matrix \( \tilde{Q}(s) \), \( \{ \tilde{Q}(s) \} \mathbf{k} = 0 \) which contradicts the assumption that rank \( \{ \tilde{h}(s) \} = m \).

With \( \mathbf{G} \) nonsingular, let

\[
\psi(s) = \begin{bmatrix} \mathbf{z}E - I + BF & \mathbf{B} \mathbf{G} \\ \mathbf{C'} & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{z}E - I & \mathbf{B} \mathbf{G} \\ \mathbf{C'} & 0 \end{bmatrix}
\]

Then, it can be verified that \( \psi(s) = \text{const.} \cdot | \mathbf{C'(s)} | \). The zeros of \( \psi(s) \) are known as the system zeros of eqn. 3 and are independent of \( \mathbf{F} \) and \( \mathbf{G} \). Now, Theorem 3 shows that if the zeros of \( | \mathbf{C'(s)} | \) are unstable, the decoupled system will be internally unstable. To avoid such undesirable pole-zero cancellation as much as possible, the unstable system zeros have to appear as those of the numerator polynomials in the decoupled system. The following result is as general as possible in this direction.

**Theorem 4**

\( (C, E, B) \) can be decoupled in the form

\[
H_{ij}(\mathbf{F}, \mathbf{G}, \mathbf{s}) = \frac{\beta_{i,j} s^{-m} + \cdots + \beta_{i,m} s^{-1}}{\hat{\alpha}_{i,j} s^{-m} + \cdots + \hat{\alpha}_{i,m} s^{-1} + \cdots + \hat{\alpha}_{i,0} s^{-1}}
\]

\[
\hat{\alpha}_{i,j} n_i(s) \quad (i = 1, 2m)
\]

(47)

where \( n_i(s) \) and \( h_i(s) \) are the coprime polynomials in \( s^{-1} \) and \( \beta_{i,0} \neq 0 \) for \( i = 1, 2, \ldots, m \) if and only if there exists an appropriate \( m \times n \) matrix \( \mathbf{K} \) satisfying

(a) \( \mathbf{C'(S)} = I + BF \) is decoupled with the integrator decoupling indices \( \{ \delta_1, \ldots, \delta_m \} \).

In the above, the coefficients of \( h_i(s) \) can be chosen arbitrarily and the integrator decoupling indices \( \{ \sigma_1, \ldots, \sigma_m \} \) and \( \{ \delta_1, \ldots, \delta_m \} \) of \( (C, E, B) \) and \( (K, E, B) \), respectively, are related by

\[
\delta_i = \sigma_i + m_i \quad (i = 1, 2, \ldots, m)
\]

(48)

Proof

Note first that through the co-ordinate transformation \( \mathbf{T} \), the conditions (a) and (b) can equivalently be stated as

(i) \( \mathbf{C'(IS)} = I + BF \) is decoupled with the integrator decoupling indices \( \{ \delta_1, \ldots, \delta_m \} \).

(ii) \( (K, E, B) \) is integrator decouplable with the decoupling indices \( \{ \delta_1, \ldots, \delta_m \} \).

This is seen from the chain of equations resulting from lemma 1:

\[
(-3) \mathbf{C'(IS)} - D_1^{-1} B_0 = (I - E_0(s)SI - D_0^{-1} B_0)
\]

\[
= \text{diag} [n_1(s) \ldots n_m(s)] K(sI - I^{-1})^{-1}
\]

\[
\times B[I - E_0(s)SI - D_0^{-1} B_0]
\]

To show the only if part, let \( \mathbf{H}(\mathbf{F}, \mathbf{G}, \mathbf{s}) \) be decoupled as eqn. 47. Then, arguing as before, we have on the ith row

\[
(-3) \mathbf{C'(IS)} - D_1^{-1} B_0 = \text{diag} [n_1(s) \ldots n_m(s)] K(sI - I^{-1})^{-1}
\]

\[
\times B[I - E_0(s)SI - D_0^{-1} B_0]
\]

(49)

Let

\[
\tilde{\mathbf{h}}(s) = \mathbf{C'(IS)} - D_1^{-1} B_0
\]

Then let the ith row \( \tilde{\mathbf{h}}(s) \) of \( \mathbf{K} \) be defined by

\[
\tilde{\mathbf{h}}(s) = \text{diag} [n_1(s) \ldots n_m(s)] K(sI - I^{-1})^{-1}
\]

(50)

for some polynomial \( \tilde{\mathbf{h}}(s) \). Then let the ith row \( \tilde{\mathbf{K}} \) of \( \mathbf{K} \) be defined by

\[
\tilde{\mathbf{K}} = \text{diag} [n_1(s) \ldots n_m(s)] K(sI - I^{-1})^{-1}
\]

(51)

This \( \mathbf{K} \) obviously satisfies (i). Moreover substituting (i) into eqn. 49 and using theorem 2, it follows that \( \mathbf{K} \) satisfies (ii) and the coefficients \( \{ \alpha_i \} \) of \( h_i(s) \) can be chosen arbitrarily. To see eqn. 48, note that, by eqn. 50, the following identity is valid:

\[
\text{eqn. 48}
\]

\[
\delta_i \mathbf{K} \quad \text{hence } R(C^*) = R(K^*) = R(F_0). \text{ Consequently, under the condition of eqn. 50, } (C, E, B) \text{ can be decoupled with the}
\]


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decoupling indices \( \{ \delta_1, \ldots, \delta_m \} \) if and only if \((C, E, B)\) can be decoupled with the decoupling indices \( \{ r_1, \ldots, \delta_m \} \). This completes the proof of the only if part. The if part now follows easily.

5 Examples

The following system is considered having \( n = 3 \) and \( m = 2 \)

\[
\begin{bmatrix}
1 & 0 & 1 \\
2 & 1 & 2 \\
0 & 1 & 0
\end{bmatrix} \dot{x} = x + \begin{bmatrix} 1 & 1 \\
0 & 1 \\
0 & 0
\end{bmatrix} u
\]

\[
y = \begin{bmatrix} 1 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix} x
\]

where, for simplicity, it is assumed that \( \sigma = 0 \) and eqn. 52 is already described in controller form. From eqn. 52, it is observed that

\[
E_0 = \begin{bmatrix} 1 & 0 & 1 \\
2 & 1 & 2
\end{bmatrix}
\]

\[
B_0 = \begin{bmatrix} 1 \\
0 \\
1
\end{bmatrix}
\]

\[
B_1 = \begin{bmatrix} 1 \\
0 \\
1
\end{bmatrix}
\]

\[
D_0 = \begin{bmatrix} 0 & 0 \\
0 & 0 \\
0 & 1
\end{bmatrix}
\]

\[
\mu_1 = 1, \quad \mu_2 = 2
\]

\[
\rho_1 = \mu_1 = 1, \quad \rho_2 = \mu_1 + \mu_2 = 3
\]

5.1 Case 1 (integrator decoupling)

Let us first check whether the system (eqn. 52) can be integrator decoupled by the state feedback. For this, calculate \( r_i \) \((i = 1, 2)\) and the matrix \( C^* \) in eqns. 34 and 35, respectively:

\[
r_1 = \min \{ k \geq 1 | [c_{11}, c_{12}, \ldots, c_{1k}] \neq 0 \} = 1
\]

\[
r_2 = \min \{ k \geq 1 | [c_{21}, c_{22}, \ldots, c_{2k}] \neq 0 \} = 2
\]

\[
C^* = \begin{bmatrix} 1 & 1 \\
0 & 1
\end{bmatrix}
\]

On the other hand

\[
E_0 = \begin{bmatrix} 1 & 1 \\
2 & 2
\end{bmatrix}, \quad \text{rank } E_0 = q = 1
\]

Now, the submatrix \( C_2 = [1, 1] \) comprising the \( i_1 = 1 \) row of \( C^* \) verifies eqn. 30. Thus, the condition in theorem 1 is satisfied and the system (eqn. 52) can be decoupled with the integrator decoupling indices \( \sigma_1 = r_1 = 1 \) and \( \sigma_2 = r_2 - 1 = 1 \). Thus a gain matrix \( G \) given by eqn. 32 is calculated as

\[
G = \begin{bmatrix} 1 & 1 \\
2 & 2
\end{bmatrix} \begin{bmatrix} 1/2 & 0 \\
0 & 1/2
\end{bmatrix}
\]

\[
= \begin{bmatrix} 1 & 0 \\
2 & 1
\end{bmatrix}
\]

where we set \( J = [0 \ 1]' \). To obtain \( F_0 \) from eqn. 33, calculate \( B^* \) and \( C^* \) using eqns. 33a and 33b:

\[
B^* = \begin{bmatrix} 0 & 0 \\
0 & 0
\end{bmatrix}
\]

\[
C^* = \begin{bmatrix} 1 & 0 \\
0 & 1
\end{bmatrix}
\]

Substituting eqns. 53 and 54 into eqn. 33 yields

\[
F_0 = \begin{bmatrix} 1 & 0 \\
0 & 1
\end{bmatrix}
\]

so that

\[
F = B_1^{-1} F_0 = \begin{bmatrix} 1 & -1 \\
0 & 1 \\
0 & 0
\end{bmatrix}
\]

\[
G = B_1^{-1} G = \begin{bmatrix} -1 & -1 \\
2 & 1
\end{bmatrix}
\]

are obtained. With these \( F \) and \( G \) used in the control law (eqn. 5), the closed-loop transfer matrix becomes \( H(F, G, S) = \text{diag} [s^{-1}, s^{-1}] \).

5.2 Case 2 (pole assignment)

Here, we consider the problem of finding \((F, G)\) such that the system (eqn. 52) is decoupled with the poles specified as

\[
H(F, G, S) = \text{diag} [(s + 1)^{-1}, (s + 2)^{-1}]
\]

Observe that in this case, \( \sigma_1 = 1, \sigma_2 = 2, \sigma_1 = 1 \) and \( \sigma_2 = 1 \). Therefore, \( B^{**} \) and \( C^{**} \) in eqn. 41 are calculated as

\[
B^{**} = \begin{bmatrix} 0 \\
0 \\
2
\end{bmatrix}
\]

\[
C^{**} = \begin{bmatrix} 1 & 1 \\
0 & 1
\end{bmatrix}
\]

and \( F_0 \) in eqn. 44 as

\[
F_0 = \begin{bmatrix} 2 & 0 \\
2 & 3 \\
2 & 2
\end{bmatrix}
\]

Consequently,

\[
F = B_1^{-1} F_0 = \begin{bmatrix} 0 & -3 & -1 \\
2 & 3 & 2
\end{bmatrix}
\]

\[
G = B_1^{-1} G = \begin{bmatrix} -1 & -1 \\
2 & 1
\end{bmatrix}
\]

where \( G \) remains unchanged. By direct calculation, the validity of eqn. 55 can be checked.

6 Conclusions

The decoupling problem for the descriptor systems by means of state feedback has been studied using trans-
formations that lead to the descriptor standard form and the controller form. The necessary and sufficient conditions for the pole assignment while decoupling are given in terms of the solvability of the matrix equation \( C_z^{-1} E_z \). Furthermore, when the conditions are satisfied, the decoupling pair \((F, G)\) is explicitly obtained. Although the conditions are characterised in terms of the system matrices transformed in controller form, finding simple characterisation of the conditions without using the controller form would be difficult.

7 References

7. CHRISTODOULOU, M.A.: 'Decoupling in the design and synthesis of singular systems', Automatica, 1986, 22, pp. 245-249