Visibility: Finding the Staircase Kernel in Orthogonal Polygons

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We consider the problem of finding the staircase kernel in orthogonal polygons, with or without holes, in the plane. Orthogonal polygon is a simple polygon in the plane whose sides are either horizontal or vertical. We generalize the notion of visibility in the following way: We say that two points $a$ and $b$ in an orthogonal polygon $P$ are visible to each other via staircase paths if and only if there exist an orthogonal chain connecting $a$ and $b$ and lying entirely in the interior of $P$. Furthermore, the orthogonal chain should have the property that the angles between the consecutive segments in the chain are either $+90^\circ$ or $-90^\circ$, and these should alternate along the chain. There are two principal types of staircases, NW-SE and NE-SW. The notion of staircase visibility has been studied in the literature for the last three decades. Based on this notion we can generalize the notion of star-shapedness. A polygon $P$ is called star-shaped under staircase visibility, or simply $s$-star if and only if there is nonempty set of points $S$ in the interior of $P$, such that any point of $S$ sees any point of $P$ via staircase path. The largest such set of points is called the staircase kernel of $P$ and denoted $\text{ker } P$. Our work is motivated by the work of Breen [1]. She proves that the staircase kernel of an orthogonal polygon without holes is the intersection of all maximal orthogonally convex polygons contained in it. We extend Breen's results for the case when the orthogonal polygon has holes. We prove the necessary geometric properties, and use them to derive a quadratic time, $O(n^2)$ algorithm for computing the staircase kernel of an orthogonal polygon with holes, having $n$ vertices in total, including the holes' vertices. The algorithm is based on the plane sweep technique, widely used in Computational Geometry [2]. Our result is optimal in the case of orthogonal polygon with holes, since the kernel (as proven) can consist of quadratic number of disjoint regions. In the case of polygon without holes, there is a linear time algorithm by Gewali [3] that is specific to the case of a polygon without holes. We present examples of our algorithm's results.

Keywords: computational geometry, polygons with holes, s-stars
Introduction and Basic Definitions

The problem of visibility in polygons is well researched in computational geometry. The well known Art Gallery Problem and its variations [2] is the base problem. To begin, consider a simply connected polygon \( P \) in the plane, \( P \) is convex if for every point \( q \) in \( P \), \( q \) is visible from each other point in \( P \). \( P \) is non-convex if there exist points \( p \) and \( q \) in \( P \) such that \( p \) is not visible from \( q \) via a line segment that lies entirely in the interior of \( P \). Thus not all pairs of points in \( P \) have direct visibility. The notion of non-convexity also provides the possibility that \( P \) is star-shaped. A polygon \( P \) is star-shaped, if there exists a point \( p \) such that for each point \( q \) in \( P \) the line segment \( pq \) lies entirely within \( P \). Since there exists such a point \( p \) we want to find the core set of \( p \), the set of all such points that see all other points in \( P \). The core set of all points \( p \) will be referred to as the kernel of \( P \). The kernel of a polygon is the intersection of all its interior half-planes, an algorithm to find the kernel was constructed to run in \( O(n) \) time. However, not all polygons are star-shaped. We can define similar notions of visibility for the restricted class of orthogonal polygons. A polygon \( P \) in the plane is orthogonal if its edges are parallel to the coordinate axes, thus its edges meet at right-angles, and the interior angle at each vertex is either 90° or 270°. The orthogonal polygons, also called rectilinear polygons have received a lot of attention in the Computational Geometry literature due to their applicability to planar layouts and many problems in automated motion planning. We introduce visibility via orthogonal paths as follows. A polygonal path in \( \mathbb{R}^2 \) from point \( p \) to point \( q \) is referred to as an orthogonal path if its edges are parallel to the coordinate axes and alternate in direction, and thus \( p \) "sees" \( q \). Trivially, from the definition, every simply connected orthogonal polygon is convex under this notion of visibility. Now, consider the restriction that the can only travel North West/South East or North East/South West. Such a path is called staircase path. This restriction does not guarantee that every orthogonal polygon is convex, refer to the polygon in Figure 1, there is a pair of points that are not visible to each other via staircase path. However a question now arises, is the orthogonal polygon star-shaped? Is there a point \( p \) that sees all other points via restricted staircases, which will now be referred to as staircase paths? An orthogonal polygon \( P \) is star-shaped via staircase paths, or s-star if there is a point \( p \) in \( P \) that sees every other point of \( P \) via staircase paths. This gives rise to the question of this research, how to find the core sets of all such points \( p \), which will be referred to as the staircase kernel of \( P \). The polygon in Figure 1 is not star-shaped under the normal definition, but is star-shaped under the definition of staircase visibility.

Figure 1. A Polygon that is s-star

![Polygon that is s-star](image-url)
Previous Work

There have been several papers on the subject of staircase visibility, and orthogonal polygons. The question was studied from the point of view of deciding whether a polygon is s-star, and if not whether it can be decomposed into s-stars. The latter question gives rise to the approach of finding the visibility graph of the decomposition and finding the staircase kernel using the visibility graph. This approach is used by Motwani et al. [4] to obtain quadratic, $O(n^2)$ algorithm for the problem in case of simply connected orthogonal polygon with holes. Interesting geometric properties were proved by Breen in her 1992 paper [1]. She characterizes the staircase kernel through maximal orthogonally convex (sub)polygons. Here we mention few of her most important results:

**Lemma 1.** If $P$ is an orthogonal polygon, then $P$ contains finitely many maximal orthogonal convex polygons.

Moreover, every orthogonally convex polygon in $P$ lies in a maximal orthogonally convex polygon in $P$.

**Lemma 2.** Let $x, y, z$ be points in $P$, and let $\lambda_1, \lambda_2, \lambda_3$ be staircase paths joining $x$ to $y$, $y$ to $z$, and $x$ to $z$, respectively. Then the bounded region $T$ determined by $\lambda = \lambda_1 \cup \lambda_2 \cup \lambda_3$ is an orthogonally convex polygon.

**Theorem 1.** Let $P$ be a simply connected orthogonal polygon which is star-shaped via staircase paths. Then

(1) the staircase kernel of $P$ is the intersection of all maximal orthogonal convex polygons in $P$ and (2) is again an orthogonally convex polygon.

We draw upon Breen’s work, since her results are of highly algorithmic nature, although she was not concerned with the algorithmic aspects or applications.

Further, there is one more relevant result on the subject. In [3], Gewali gives linear time algorithm, $O(n)$, for the staircase kernel of an orthogonal polygon without holes. Although this result appeared three years later than Breen’s work, it does not seem to refer to it or use any ideas. The author uses ad-hoc approach to derive the algorithmic result.

Our Approach

We extend the result of Breen to simply connected orthogonal polygons with holes. Then, we devise an algorithm for finding the kernel of an s-star based on the line sweep technique. The algorithm is robust in the sense that it recognizes the situations when the polygon is not s-star, i.e. the kernel is empty. It also recognizes the situation when a polygon with holes has connected kernel. Normally, the kernel contains quadratic number, $O(n^2)$, of disjoint components as shown in Figure 2. The kernel is shown in green. Thus, any quadratic time algorithm that computes it is worst-case optimal.
Key to our approach, besides the use of line sweep, is the fact that we decompose the input polygon into rectangles. The main motivation for this is that decomposing the polygon into maximal orthogonally convex subpolygons might not be algorithmically nice. Further, the maximally orthogonal subpolygons will be in general complex (as complex as linear in the size of the input) and their intersection, therefore much more complex than intersecting two rectangles, which is constant time operation, $O(1)$. The correctness of our approach lies upon the following two geometric properties.

**Lemma 3** (Parallel Corridor Lemma). If a horizontal (or vertical) line intersects an orthogonal polygon $P$, and the intersection consists of two or more disjoint line segments, then none of the regions immediately adjacent to the line is part of the staircase kernel.

**Proof.** The situation is illustrated in Figure 3. The fact that none of the regions adjacent to the sweep line is part of the kernel follows directly from the violation of the definition of staircase visibility, as shown in the figure. As proven, the kernel, or any part of it when it consists of disjoint parts has to be orthogonally convex. Here the horizontal (or vertical, in the case of vertical line) convexity is violated.

**Lemma 4** (Pocket lemma). If a horizontal (or vertical) line intersects an orthogonal polygon $P$ such that the intersection is non-convex, and the next (previous) position of the sweep line is also non-convex, but with smaller number of disjoint segments, then the entire part of the polygon below the second line does not contain any part of the kernel.

**Proof.** The situation is illustrated in Figure 4. What the lemma says is that if we perform a horizontal line sweep, top-down and we are going through parallel corridors, they are not part of the kernel, as proven in Lemma 3. Here, a stronger statement is true. Consider the region (or regions) that is (are) becoming closed, while some other regions are still extending down. We call such a closing region a pocket. In the figure, the pocket is the leftmost of the three parallel corridors. It is clear that any point in the pocket cannot be visible via staircase paths of either type from any point below the sweep line past the "bottom" of the pocket. Thus, the part of the polygon below the "bottom" of the pocket cannot contain any points from the kernel of the polygon.
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Figure 3. Parallel Corridor Lemma, Violation of the Staircase Visibility Conditions

Figure 4. Pocket Lemma, Violation of the Staircase Visibility Condition

Note that the Pocket Lemma is more versatile than it seems. It can be applied in either direction, i.e. if we get another parallel corridor, while sweeping through parallel corridors, then nothing previously seen sees that pocket, therefore we have no part of the kernel in the already swept part of the polygon. Algorithmically, one can do two passes of sweeping in the same direction - one top-down, and one bottom-up. Same applies to the vertical
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sweeps with respect to both lemmas.

Now, we have a solid basis to formulate our algorithmic approach.

The Algorithm

Algorithm StaircaseKernel($P$)
Input: An orthogonal polygon $P \in \mathbb{R}^2$, possibly with holes.
Output: List $K$ containing the staircase kernel of $P$.
1. Build two lists $EX$, $EY$ containing the horizontal and vertical edges of $P$, respectively, sorted.
2. Use $EX$ to perform horizontal line sweep of $P$ in the following way:
   3. if the next edge $e$ in $EX$ is a single edge
      4. then close the current region(s), add them to the list $XR$
      5. if there is more than one region
         6. then mark each of them as impossible
         7. else mark the current region as possible
      8. if the edge $e$ is not adjacent to any of the current regions
         9. then exit the vertical sweep, go to 12.
      10. else start a new region with a top edge $e$
      11. else perform steps 4-10 for each of the edges $e_i$ with same $x$-coordinate simultaneously
   12. Use $EY$ to perform vertical line sweep of $P$ in the following way:
      13. if the next edge $e$ in $EY$ is a single edge
         14. then close the current region(s), add them to the list $YR$
         15. if there is more than one region
             16. then mark each of them as impossible
             17. else mark the current region as possible
         18. if the edge $e$ is not adjacent to any of the current regions
             19. then exit the vertical sweep, go to 22.
         20. else start a new region with a left edge $e$
      21. else perform steps 14-20 for each of the edges $e_i$ with same $y$-coordinate simultaneously
   22. Obtain the list $HR$ from the list $XR$ by eliminating all the impossible regions from $XR$
   23. Obtain the list $VR$ from the list $YR$ by eliminating all the impossible regions from $YR$
   24. if either $HR$ or $VR$ is empty, report $K$ is empty, exit
   25. Obtain the list $K[i, j]$ by intersecting each entry $i$ in $HR$ with each entry $j$ from $VR$
   26. if $K$ is empty, report and exit
   27. for $i := 1$ to $|HR|$ do
      28. for $j := 1$ to $|VR| - 1$ do
         29. if $K[i, j]$ and $K[i, j + 1]$ are disjoint
            30. then if $P$ has no holes report $K$ is empty, exit
            31. else skip to the next iteration on $i$
         32. $K[i, j + 1] := K[i, j] \cup K[i, j + 1]$
   33. report $K$
Analysis of the Algorithm

**Theorem 2.** The algorithm $\text{StaircaseKernel}(P)$ computes correctly the staircase kernel of an orthogonal polygon $P$, possibly with holes, in quadratic time and space, $O(n^2)$, where $n$ is the total number of vertices in $P$, including holes' vertices.

**Proof.** The correctness of the computation of the kernel follows from the theoretical considerations earlier in the paper, in particular, Theorem 1 and Lemmas 1 through 4. The timing analysis follows. Computing the sorted lists of horizontal and vertical edges, $EX$ and $EY$ in line 1 takes $O(n \log n)$ time, since each of them is of at most $O(n)$ size. The horizontal line sweep, lines 2-11, and the vertical line sweep, lines 12-21 take $O(n \log n)$ time each. This is due to the fact that we have to use balanced binary search trees as a status structure. This is done in the standard way shown in [2], with one exception. When we encounter edges that are sharing the same $x$ (or $y$) coordinate, we have to deal with them simultaneously, due to the nature of our problem here.

However, it is clear that the new regions created are at most one per new edge, all the current regions are closed and marked/labeled correctly as per Lemmas 3 and 4. Further, because of charging each new region to its top/left edge, the size of the lists that we create, $HR$ and $YR$ is at most linear in $n$. Lines 22 and 23 clearly take linear time, $O(n)$. Line 25 takes quadratic time, $O(n^2)$. The fragment in lines 27-32 takes quadratic time in total, $O(n^2)$, due to the fact that each of the operations inside the two nested loops of size $O(n)$ each is performed in constant time: union or checking for intersection of rectangles. Note that here we take union and check for intersection of two rectangles that are not in general position with respect to each other. Here the two rectangles are part of the same horizontal strip given by the initial $i$-th entry in $HR$. Line 33 takes at most $O(n^2)$ time. Finally, lines 24 and 26 are performed in constant time, $O(1)$.

**Examples**

In this section we present experimental results of our algorithm on polygons covering the most typical cases. The first example, the H polygon shows an orthogonal polygon with holes that has nonempty kernel. The second example, the so-called amoeba shows another orthogonal polygon with holes that has non-empty kernel. The purpose of this example is to illustrate the elimination of the "false" parts of the kernel. The third example, the double E polygon is a polygon that has no holes, but is not an s-star, i.e. its staircase kernel is empty. Our final example, the so-called teapot is an orthogonal polygon with holes, and its kernel consists of disjoint parts.
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The H Polygon

**Figure 5.** The Horizontal Convex Regions, Vertical Convex Regions, and the Staircase Kernel of the H Polygon

The Amoeba

**Figure 6.** The Amoeba Polygon
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**Figure 7.** The Horizontal Convex Regions and Vertical Convex Regions of the Amoeba Polygon

**Figure 8.** The Intersection of the Regions and the Kernel of the Amoeba Polygon
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The Double E

Figure 9. The Double E Polygon and its Horizontal and Vertical Convex Regions

Figure 10. The Intersection of the Regions for the Double E Polygon, the Kernel is Disjoint, i.e. not an s-star
The Teapot

**Figure 11.** The Teapot Polygon and its Horizontal and Vertical Convex Regions

![Teapot Polygon and its Horizontal and Vertical Convex Regions](image1)

**Figure 12.** The Staircase Kernel of the Teapot Polygon

![Staircase Kernel of the Teapot Polygon](image2)

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