Midterm Examination (Solutions)

1. (a) Define mean, variance and standard deviation of a sample.

(b) Calculate the mean, the variance and the standard deviation of the following data:

$$16, 8, 14, 15, 13, 11, 11, 19, 10, 7$$

Solution:

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i = \frac{1}{10}(16+8+14+15+13+11+11+19+10+7) = \frac{124}{10} = 12.4$$

Thus, the mean of this data is 12.4.

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i-\bar{y})^2 = \frac{1}{9}\Big[(16-12.4)^2+(8-12.4)^2+(14-12.4)^2+(15-12.4)^2+(13-12.4)^2+$$

$$+(11-12.4)^2+(11-12.4)^2+(19-12.4)^2+(10-12.4)^2+(7-12.4)^2\Big] =$$

$$= \frac{1}{9}\big(3.6^2+4.4^4+1.6^2+2.6^2+0.6^2+1.4^4+1.4^2+6.6^2+2.4^2+5.4^2\big) = \frac{124.4}{9} = 13.82$$

The variance of the given data is 13.82 (rounded to two decimal places), and the standard deviation is the square root of the variance, 3.72 again rounded to two decimal places.

2. (a) Define the permutation, P_r^n, and the combination, C_r^n of n distinct objects taken r at a time. Derive formulae for P_r^n and C_r^n.

(b) Ten teams are playing in a basketball tournament. In the first round, the teams are randomly assigned to games 1, 2, 3, 4, and 5. In how many different ways can the teams be assigned to the games?

Solution: There are two ways to obtain this answer. First, to notice that this is a partition of the ten teams into subsets of two, so it can be done in

$$\binom{10}{2\;2\;2\;2\;2} = \frac{10!}{2!2!2!2!2!} = \frac{10!}{2^5} = \frac{3628800}{32} = 113400$$

different ways.

Second, we can consider these consecutively. The first two teams can be assigned to the first game in $\binom{10}{2} = 45$ different ways. Out of the remaining teams, the assignment for the second game can be done in $\binom{8}{2} = 28$ different ways, then the assignment to the third game can be done in $\binom{6}{2} = 15$ different ways, the assignment to the fourth game can be done in $\binom{4}{2} = 6$ different ways, and finally the assignment of the teams to the fifth game can be done in $\binom{10}{2} = 1$ only one way. Thus, we have $45 \cdot 28 \cdot 15 \cdot 6 \cdot 1 = 113400$ different ways to make the assignments.

3. (a) State and prove the multiplicative law of probability for any number k of events $A_1, A_2, \ldots, A_k$.

(b) Two events A and B are such that $P(A) = 0.2, P(B) = 0.3, P(A \cup B) = 0.4$. Find $P(\overline{A}|B)$.

Solution: Using the definition of conditional probability, we have

$$P(\overline{A}|B) = \frac{P(\overline{A}\cap B)}{P(B)}$$

Here $P(B)$ is given, we have to find $P(\overline{A}\cap B)$. One way to do it is to represent B as

$$B = (A\cap B)\cup(\overline{A}\cap B)$$

Note that this is in fact a partition of B since the sets $A \cap B$ and $\overline{A} \cap B$ are disjoint. Therefore,

$$P(B) = P(A \cap B) + P(\overline{A} \cap B) \Rightarrow P(\overline{A} \cap B) = P(B) - P(A \cap B)$$

From the inclusion-exclusion principle, we have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \Rightarrow P(B) - P(A \cap B) = P(A \cup B) - P(A) = 0.4 - 0.2 = 0.2$$

Therefore, $P(\overline{A} \cap B) = 0.2$. Finally,

$$P(\overline{A}|B) = \frac{P(\overline{A} \cap B)}{P(B)} = \frac{0.2}{0.3} = \frac{2}{3}$$

4. (a) State and prove the Law of Total Probability.

(b) A student answers a multiple-choice examination question that offers four possible answers. Suppose the probability that the student knows the answer to the question is 0.8 and the probability that the student will guess is 0.2. Assume that if the student guesses, the probability of selecting the correct answer is 0.25. If the student correctly answers a question, what is the probability that the student really knew the correct answer?

Solution: Here we have to apply the Bayes' Rule. Let the events that we consider be G - the student guesses the answer to the question and C - the student gives the correct answer. We are given that

$$P(G) = 0.2,\ P(\overline{G}) = 0.8,\ P(C|G) = 0.25$$

Further, it is obvious that $P(C|\overline{G}) = 1$, that is, if the student knows the correct answer the probability that they will give the correct answer (when not guessing) is 1. We have:

$$P(\overline{G}|C) = \frac{P(\overline{G} \cap C)}{P(C)} = \frac{P(C|\overline{G}) \cdot P(\overline{G})}{P(C|G) \cdot P(G) + P(C|\overline{G}) \cdot P(\overline{G})} = \frac{1 \cdot 0.8}{0.25 \cdot 0.2 + 1 \cdot 0.8} = \frac{0.8}{0.85} = \frac{8}{85} \simeq 0.9412$$

5. (a) Define a discrete random variable, its probability distribution and its expected value.

(b) A potential customer for an \$85,000 fire insurance policy possesses a home in an area that, according to experience, may sustain a total loss in a given year with probability of 0.001 and a 50% loss with probability 0.01. Ignoring all other partial losses, what premium should the insurance company charge for a yearly policy in order to break even on all \$85,000 policies in this area?

Solution: Let Y be the discrete random variable that represents the amount of money that insurance company will have to pay on a single policy for a home in the area. The set of possible values of Y is $\{0, 42500, 85000\}$. According to the statement, we have the following probability distribution for Y:

$$P(Y = 0) = 0.989,\ P(Y = 42500) = 0.01,\ P(Y = 85000) = 0.001$$

Thus, the expected amount of money paid on a single policy is

$$E(Y) = \sum_y y \cdot p(y) = 0 \cdot 0.989 + 42500 \cdot 0.01 + 85000 \cdot 0.001 = 0 + 425 + 85 = 510$$

The company should charge \$510 for a yearly policy on a home in this area in order to break even (i.e. make no profit).