Title: Total Curvature of a Path
Speaker: Krystyna Kuperberg, Auburn University

This talk is based on joint work with I. Bárány and T. Zamfirescu. Some results included in the PhD dissertation of Robert Ford will be mentioned also.

The *total curvature* $\tau(C)$ of a path C in $\mathbb{R}^n$ is defined as follows:

1. For a C^2 path C with arc-length parameterization $r(s)$, $\tau(C) = \int_C |r''(s)|\, ds$.
2. For a polygonal path $C = [z_0, z_1, \ldots, z_n]$, the total curvature, also known as the total turning angle, $\tau(C) = \sum_{i=1}^{n-1}(\pi - \angle z_{i-1} z_i z_{i+1})$.

The notion is easily generalized to paths that are piecewise C^2. It is known (W. Fenchel, K. Borsuk) that the total curvature of a simple closed curve is bounded from below by 2π, with the equality holding only for flat convex simple closed curves, i.e., contained in a 2-dimensional plane. J. Milnor proved that the total curvature of a knot in $\mathbb{R}^3$ is greater than 4π.

Let $\mathcal{K}$ be the set of all compact convex polyhedra in $\mathbb{R}^3$. Consider the triples (K, a, b), where $K \in \mathcal{K}$ and $a, b \in \partial(K)$, and the relation between paths $C(a, b)$ and $P(a, b)$ such that:

1. $C_{K(a,b)}$ is a path of smallest total curvature joining a and b in $\partial(K)$,
2. $P_{K(a,b)}$ is a shortest path joining a and b in $\partial(K)$.

These two paths seldom coincide and only in some special cases the paths are flat, see **[F]**.

Let $\mathcal{P} = \{P_{K(a,b)} \,|\, K \in \mathcal{K} \text{ and } a, b \in \partial(K)\}$. Define the set $\mathbf{T} \subset \mathbb{R}$ of total curvatures as

$$\mathbf{T} = \{\tau(P) \,|\, P \in \mathcal{P}\}.$$

The **Bounded Total Curvature Conjecture** asserts that $\mathbf{T}$ is bounded.

History of the problem: A.V. Pogorelov **[Po]** considered another number associated with $\mathcal{P}$. Let $P = P_{K(a,b)} = [z_0, z_1, \ldots, z_n]$. For $i = 1, \ldots, n$, let $\vec{u}_i$ be the outernormal vector of K, normal to the facet containing $[z_{i-1}, z_i]$, and let α_i be the angle between $\vec{u}_i$ and $\vec{u}_{i+1}$, the folding angle. The question posed in **[Po]** was whether the set of all sums $\mathbf{F} = \sum_{i=1}^{n-1} \alpha_i$ for $P \in \mathcal{P}$ was bounded. This question was answered in the negative by J. Pach **[Pa]**. Another example showing that $\mathbf{F}$ is not bounded is given in **[BKZ]**. A shortest path may be “spiraling” in the boundary of a very long convex body.

Pach's example gave rise to the Bounded Total Curvature Conjecture, which was stated in [**AHSV**]. One can also ask for a partial solution to the total curvature question: Is **T** bounded by some specific number, for example 2π? If P is flat, then $\tau(P) < 2\pi$. A simple example shows that this inequality is sharp. It is shown in [**BKZ**], that in general, $\tau(P)$ may exceed 2π, but it is not clear by how much.

It is shown in [**BKZ**] that the Bounded Total Curvature Conjecture is true for convex bodies that are relatively round. If a convex polyhedral body K satisfies $rB \subset K \subset B$, where B is the unit ball, then the total curvature of any shortest path on the boundary of K is less than $4\pi^2 r^{-2}$.

The method used to prove this theorem has potential for generalizations to elongated convex bodies. We will show certain estimates for such bodies. This, however, does not provide a solution to the Bounded Total Curvature Conjecture.

References

[**AHSV**] P. Agarwal, S. Har-Peled, M. Sharir, K. Varadarajan, *Approximating shortest paths on a convex polytope in three dimensions*, J.A.C.Math. **44** (1997), 557-584.

[**BKZ**] I. Bárány, K. Kuperberg, T. Zamfirescu, *Total curvature and spiraling shortest paths*, Discrete and Computational Geometry 22, 367-375.

[**Fo**] R. Ford, *Path curvatures on a convex roof*, PhD dissertation, Auburn University, 2007.

[**Pa**] J. Pach, *Folding and turning along geodesics in a convex surface*, Geombinatorics, **7** (1997) 61-65.

[**Po**] A.V. Pogorelov, Extrinsic geometry of convex surfaces. Translations of Mathematical Monographs **35**, Amer. Math. Soc., Providence, RI, 1973.