Wiener Index of Line Graphs
and
Bounds on Gutman Index

Darko Dimitrov
Freie Universität Berlin

Workshop on Algorithmic Graph Theory
Nipissing University
19.05.2011
Overview of the talk

Wiener Index of Line Graphs

- Definitions
- Known results
- Connected graph with $\delta(G) \geq 2$
- $W(G) = W(L(G))$?
Overview of the talk

Wiener Index of Line Graphs

- Definitions
- Known results
- Connected graph with $\delta(G) \geq 2$
- $W(G) = W(L(G))$?

Bounds on Gutman index

- Related results
- The graph with minimal Gutman index
- Bounds for graphs with minimal and graphs with maximal Gutman index
Definition

\[ G = (V, E) \] a finite, simple and undirected graph
Definition

\( G = (V, E) \) a finite, simple and undirected graph

Wiener index

\[ W(G) = \sum_{u, v \in V(G)} d(u, v) \]
**Definition**

\( G = (V, E) \) a finite, simple and undirected graph

**Wiener index**

\[
W(G) = \sum_{u,v \in V(G)} d(u, v)
\]

Harold Wiener 1947
edge-Wiener index

\[ W(G) = \sum_{e,e' \in E(G)} d(e, e') \]
Definitions

Given a graph $G$, its line graph $L(G)$ is a graph such that

- The vertices of $L(G)$ are the edges of $G$; and
- Two vertices of $L(G)$ are adjacent if and only if their corresponding edges in $G$ share a common endvertex.
Given a graph $G$, its line graph $L(G)$ is a graph such that

- The vertices of $L(G)$ are the edges of $G$; and
- Two vertices of $L(G)$ are adjacent if and only if their corresponding edges in $G$ share a common endvertex.

Let $G$ be a connected graph. Then the edge-Wiener index of $G$ is

$$W_e(G) = \sum_{\{e, e'\} \subseteq E(G)} d(e, e')$$

where the distance between two edges is the distance between the corresponding vertices in the line graph of $G$. 
Relation between $W(G)$ and $W(L(G))$
Relation between $W(G)$ and $W(L(G))$

**Theorem 1** (Buckley, 1981). *For every tree $T$*,

\[ W(L(T)) = W(T) - \binom{n}{2} \]
Relation between $W(G)$ and $W(L(G))$

**Theorem 1** (Buckley, 1981). For every tree $T$,

$$W(L(T)) = W(T) - \binom{n}{2}.$$ 

**Theorem 2** (Gutman, Pavlović, 1997). If $G$ is a connected graph with $n$ vertices and $m$ edges, then

$$W(L(G)) \geq W(G) - n(n - 1) + \frac{1}{2}m(m + 1).$$
Theorem 3 (Cohen, D., Krakovski, Škrekovski, Vukašinović, 2010). Let $G$ be a connected graph with $\delta(G) \geq 2$. Then,

$$W(G) \leq W(L(G)).$$

Moreover, equality holds only for cycles.
Theorem 3 (Cohen, D., Krakovski, Škrekovski, Vukašinović, 2010). Let $G$ be a connected graph with $\delta(G) \geq 2$. Then,

$$W(G) \leq W(L(G)).$$

Moreover, equality holds only for cycles.

Proof.

$G$ is a cycle
\[ W(L(G)) = \sum_{e, e' \in E(G) \atop e \neq e'} d(e, e') \]
\[ W(L(G)) = \sum_{e, e' \in E(G)} d(e, e') \]

\[ \geq \frac{1}{4} \sum_{e = uv \in E(G)} \left( d(u, u') + d(u, v') + d(v, u') + d(v, v') \right) - \]
\begin{align*}
W(L(G)) &= \sum_{e, e' \in E(G)} d(e, e') \\
&\geq \frac{1}{4} \sum_{e = uv \in E(G)} \left( d(u, u') + d(u, v') + d(v, u') + d(v, v') \right) \\
&= \frac{1}{4} \left[ \sum_{u, v \in V(G), \ uv \not\in E(G)} d(u) d(v) d(u, v) + \sum_{u, v \in V(G), \ uv \in E(G)} \left( d(u) d(v) - 1 \right) d(u, v) \right]
\end{align*}
\[ W(L(G)) - W(G) \geq \frac{1}{4} \left[ \sum_{u,v \in V(G)} d(u)d(v)d(u,v) + \sum_{u,v \in V(G)} \left( d(u)d(v) - 1 \right) \right] - \sum_{u,v \in V(G)} d(u,v) \]
\[ W(L(G)) - W(G) \geq \frac{1}{4} \left[ \sum_{u,v \in V(G)} d(u)d(v)d(u,v) + \sum_{u,v \in V(G)} \left( d(u)d(v) - 1 \right) \right] - \sum_{u,v \in V(G)} d(u,v) \]

\[ = \frac{1}{4} \left[ \sum_{u,v \in V(G)} (d(u)d(v) - 4)d(u,v) + \sum_{u,v \in V(G)} (d(u)d(v) - 5) \right] \]
\[ W(L(G)) - W(G) \geq \frac{1}{4} \left[ \sum_{u,v \in V(G), \ uv \notin E(G)} d(u)d(v)d(u,v) + \sum_{u,v \in V(G), \ uv \in E(G)} \left( d(u)d(v) - 1 \right) \right] - \sum_{u,v \in V(G)} d(u,v) \]

\[ = \frac{1}{4} \left[ \sum_{u,v \in V(G), \ uv \notin E(G)} \left( d(u)d(v) - 4 \right)d(u,v) + \sum_{u,v \in V(G), \ uv \in E(G)} \left( d(u)d(v) - 5 \right) \right] \]

Let \( G_2 \) be the graph induced by the vertices of degree two in \( G \).
\[ W(L(G)) - W(G) \geq \frac{1}{4} \left[ \sum_{u,v \in V(G)} d(u)d(v)d(u,v) + \sum_{u,v \in V(G)} \left( d(u)d(v) - 1 \right) \right] - \left( \sum_{u,v \in V(G)} d(u,v) \right) \]

\[ = \frac{1}{4} \left[ \sum_{u,v \in V(G)} \left( d(u)d(v) - 4 \right)d(u,v) + \sum_{u,v \in V(G)} \left( d(u)d(v) - 5 \right) \right] \]

Let \( G_2 \) be the graph induced by the vertices of degree two in \( G \)

\[ \sum_{u,v \in V(G_2)} \left( d_{G_2}(u)d_{G_2}(v) - 4 \right)d_{G_2}(u,v) = 0 \]
$W(L(G)) - W(G) \geq \frac{1}{4} \left[ \sum_{u,v \in V(G), \ uv \not\in E(G)} d(u)d(v)d(u,v) + \sum_{u,v \in V(G), \ uv \in E(G)} (d(u)d(v) - 1) \right] - \sum_{u,v \in V(G)} d(u,v)$

$$= \frac{1}{4} \left[ \sum_{u,v \in V(G), \ uv \not\in E(G)} (d(u)d(v) - 4)d(u,v) + \sum_{u,v \in V(G), \ uv \in E(G)} (d(u)d(v) - 5) \right]$$

Let $G_2$ be the graph induced by the vertices of degree two in $G$

$$\sum_{u,v \in V(G_2), \ uv \not\in E(G_2)} \left( d_{G_2}(u)d_{G_2}(v) - 4 \right) d_{G_2}(u,v) = 0$$

$$\sum_{u,v \in V(G_2), \ uv \in E(G_2)} \left( d_{G_2}(u)d_{G_2}(v) - 5 \right) = -|E(G_2)|$$
\[ W(L(G)) - W(G') \geq \frac{1}{4} \left[ \sum_{u, v \in V(G)} \left( d_G(u)d_G(v) - 4 \right) d_G(u, v) + \sum_{u, v \in V(G)} \left( d_G(u)d_G(v) - 5 \right) - |E(G_2)| \right] \]
\[ W(L(G)) - W(G') \geq \frac{1}{4} \left[ \sum_{u,v \in V(G)} \left( d_G(u)d_G(v) - 4 \right) d_G(u,v) + \sum_{u,v \in V(G)} \left( d_G(u)d_G(v) - 5 \right) - |E(G_2)| \right] \]

\[ d_G(u)d_G(v) - 4 \geq 2 \]
\[ W(L(G)) - W(G') \geq \frac{1}{4} \left[ \sum_{u, v \in V(G)} (d_G(u)d_G(v) - 4)d_G(u, v) + \sum_{u, v \in V(G)} (d_G(u)d_G(v) - 5) - |E(G_2)| \right] \]

\[ d_G(u)d_G(v) - 4 \geq 2 \]

\[ d_G(u)d_G(v) - 5 \geq 1 \]
\[ W(L(G)) - W(G') \geq \frac{1}{4} \left[ \sum_{u,v \in V(G)} \left( d_G(u)d_G(v) - 4 \right) d_G(u,v) + \sum_{\{u,v\} \not\subseteq V(G_2)} \left( d_G(u)d_G(v) - 5 \right) - |E(G_2)| \right] \]

\[ d_G(u)d_G(v) - 4 \geq 2 \]

\[ d_G(u)d_G(v) - 5 \geq 1 \quad \Rightarrow \quad \sum_{\{u,v\} \not\subseteq V(G_2)} \left( d_G(u)d_G(v) - 5 \right) \geq |V(G_2)| \]
\[ W(L(G)) - W(G') \geq \frac{1}{4} \left[ \sum_{\{u,v\} \not\subseteq V(G)} \left( d_G(u)d_G(v) - 4 \right) d_G(u,v) + \right. \\
\left. \sum_{\{u,v\} \not\subseteq V(G)} \left( d_G(u)d_G(v) - 5 \right) - |E(G_2)| \right] \]

\[
d_G(u)d_G(v) - 4 \geq 2
\]

\[
d_G(u)d_G(v) - 5 \geq 1 \quad \Rightarrow \quad \sum_{\{u,v\} \not\subseteq V(G)} \left( d_G(u)d_G(v) - 5 \right) \geq |V(G_2)|
\]

\[
|V(G_2)| \geq |E(G_2)| \quad \text{for any graph of maximum degree 2}
\]
\[
W(L(G)) - W(G') \geq \frac{1}{4} \left[ \sum_{\{u,v\} \not\subseteq V(G_2)} \left( d_G(u)d_G(v) - 4 \right) d_G(u,v) + \sum_{u,v \in V(G) \atop \{u,v\} \not\subseteq V(G_2) \atop uv \not\in E(G)} \left( d_G(u)d_G(v) - 5 \right) - |E(G_2)| \right]
\]

\[
d_G(u)d_G(v) - 4 \geq 2
\]

\[
d_G(u)d_G(v) - 5 \geq 1 \Rightarrow \sum_{u,v \in V(G) \atop \{u,v\} \not\subseteq V(G_2) \atop uv \in E(G)} \left( d_G(u)d_G(v) - 5 \right) \geq |V(G_2)|
\]

\[
|V(G_2)| \geq |E(G_2)| \text{ for any graph of maximum degree } 2
\]

\[
W(L(G')) - W(G) > 0
\]
\[ W(G) = W(L(G)) \]
$W(G) = W(L(G))$
\[ W(G) = W(L(G)) \]

Cycles

\[ W(G') = W(L(G)) \]

Other unicycle graphs

\[ W(G) > W(L(G)) \]
\[ W(G) = W(L(G)) \]

Cycles  
\[ W(G') = W(L(G)) \]

Other unicycle graphs  
\[ W(G') > W(L(G)) \]

Bicyclic graphs  
\[ W(G') < W(L(G')) \]
\[ |V(G)| < 6 \]

\[ W(G') = W(L(G)) \]

\[ W(G') > W(L(G)) \]
Related results

**Theorem 4** (Dobrynin and Mel’nikov, 2005). *There are infinite families of graphs of girth three and four with the property*

\[ W(L(G)) = W(G) \]
Related results

**Theorem 4** (Dobrynin and Mel’nikov, 2005). *There are infinite families of graphs of girth three and four with the property*

\[ W(L(G)) = W(G) \]

*.

**Problem 1** (Dobrynin and Mel’nikov, 2005). *Is it true that for every integer \( g \geq 5 \), there exists a graph \( G \neq C_g \) of girth \( g \), for which \( W(G) = W(L(G)) \)?
Theorem 5 (Cohen, D., Krakovski, Škrekovski, Vukašinović, 2010). For every positive integer \( g_0 \), there exists \( g \geq g_0 \) such that there are infinitely many graphs \( G \) of girth \( g \) satisfying \( W(G) = W(L(G)) \).
**Theorem 5** (Cohen, D., Krakovski, Škrekovski, Vukašinović, 2010). *For every positive integer \( g_0 \), there exists \( g \geq g_0 \) such that there are infinitely many graphs \( G \) of girth \( g \) satisfying \( W(G) = W(L(G)) \).*

---

**Theorem 6.** *For every non-negative integer \( h \), there exist infinitely many graphs \( G \) of girth \( g = h^2 + h + 9 \) with \( W(L(G)) = W(G) \).*
Lemma 1. For integers, $k, p, q \geq 1$, let $G = \Phi(k, p, q)$ with girth $g = 2k + 1$. Then,

$$W(L(G)) - W(G) = \frac{1}{2} \left( g^2 + (p - q)^2 + 5(p + q - 3) - 2g(p + q - 3) \right).$$
Theorem 7. For every non-negative integer $h$, there exist infinitely many graphs $G$ of girth $g = h^2 + h + 9$ with $W(L(G)) = W(G)$. 
Theorem 7. For every non-negative integer $h$, there exist infinitely many graphs $G$ of girth $g = h^2 + h + 9$ with $W(L(G)) = W(G)$.

Claim 1. Let $a_0, a_1, k$, such that $W(L(\Phi(k, a_0, a_1))) = W(\Phi(k, a_0, a_1))$ and $a_0 < a_1$. Then, from $a_0$ and $a_1$, we can build an infinite strictly increasing sequence $a_0, a_1, a_2, \ldots$ of integers such that for every $n \geq 0$, $W(L(\Phi(k, a_n, a_{n+1}))) = W(\Phi(k, a_n, a_{n+1}))$. 
\[ D_n = W(L(\Phi(k, a_n, a_{n+1}))) - W(\Phi(k, a_n, a_{n+1})) \]
\[ = \frac{1}{2}g^2 - ga_n - ga_{n+1} + \frac{1}{2}a_n^2 - a_na_{n+1} + \]
\[ \frac{1}{2}a_{n+1}^2 + 3g + \frac{5}{2}(a_n + a_{n+1}) - \frac{15}{2} \]
\[ = 0, \]
\[ D_n = W(L(\Phi(k, a_n, a_{n+1}))) - W(\Phi(k, a_n, a_{n+1})) \]
\[ = \frac{1}{2}g^2 - ga_n - ga_{n+1} + \frac{1}{2}a_n^2 - a_n a_{n+1} + \]
\[ \frac{1}{2}a_{n+1}^2 + 3g + \frac{5}{2}(a_n + a_{n+1}) - \frac{15}{2} \]
\[ = 0, \]

\[ D_n - D_{n+1} = (a_{n+2} - a_n)(g - \frac{1}{2}(a_{n+2} + a_n) + a_{n+1} - \frac{5}{2}) \]
\[ = 0. \]
\[ g - \frac{1}{2}(a_{n+2} + a_n) + a_{n+1} - \frac{5}{2} = 0 \]
\[ g - \frac{1}{2} (a_{n+2} + a_n) + a_{n+1} - \frac{5}{2} \] = 0

\[ a_n = c_n + p_n \]
\[ g - \frac{1}{2}(a_{n+2} + a_n) + a_{n+1} - \frac{5}{2} = 0 \]

\[ a_n = c_n + p_n \]

\[ c_n = nx + y, \text{ for } x, y \in \mathbb{R}, \text{ is the homogeneous solution} \]
\[ g - \frac{1}{2}(a_{n+2} + a_n) + a_{n+1} - \frac{5}{2} = 0 \]

\[ a_n = c_n + p_n \]

\[ c_n = nx + y, \text{ for } x, y \in \mathbb{R}, \text{ is the homogeneous solution} \]

\[ p_n = cn^2, \text{ for } c \in \mathbb{R}, \text{ is the particular solution} \]
\[ g - \frac{1}{2}(a_{n+2} + a_n) + a_{n+1} - \frac{5}{2} = 0 \]

\[ a_n = c_n + p_n \]

\[ c_n = nx + y, \text{ for } x, y \in \mathbb{R}, \text{ is the \textit{homogeneous solution}} \]

\[ p_n = cn^2, \text{ for } c \in \mathbb{R}, \text{ is the \textit{particular solution}} \]

\[ y = a_0, \ x = (\frac{5}{2} + a_1 - g - a_0) \text{ and } c = g - \frac{5}{2} \]
\[ g - \frac{1}{2}(a_{n+2} + a_n) + a_{n+1} - \frac{5}{2} = 0 \]

\[ a_n = c_n + p_n \]

\[ c_n = nx + y, \text{ for } x, y \in \mathbb{R}, \text{ is the homogeneous solution} \]

\[ p_n = cn^2, \text{ for } c \in \mathbb{R}, \text{ is the particular solution} \]

\[ y = a_0, \quad x = \left( \frac{5}{2} + a_1 - g - a_0 \right) \quad \text{and} \quad c = g - \frac{5}{2} \]

\[ a_n = \left( g - \frac{5}{2} \right)n^2 + \left( \frac{5}{2} + a_1 - g - a_0 \right)n + a_0 \]
Lemma 2. For integers, $k, p, q \geq 1$, let $G = \Phi(k, p, q)$ with girth $g = 2k + 1$. Then,

$$W(L(G)) - W(G) = \frac{1}{2} (g^2 + (p - q)^2 + 5(p + q - 3) - 2g(p + q - 3)).$$
Lemma 2. For integers, $k, p, q \geq 1$, let $G = \Phi(k, p, q)$ with girth $g = 2k + 1$. Then,

$$W(L(G)) - W(G) = \frac{1}{2} (g^2 + (p - q)^2 + 5(p + q - 3) - 2g(p + q - 3)).$$

$$g = -3 + p + q + \sqrt{24 - 11p - 11q + 4pq}.$$
Lemma 2. For integers, $k, p, q \geq 1$, let $G = \Phi(k, p, q)$ with girth $g = 2k + 1$. Then,

$$W(L(G)) - W(G) = \frac{1}{2} \left( g^2 + (p - q)^2 + 5(p + q - 3) - 2g(p + q - 3) \right).$$

$$g = -3 + p + q + \sqrt{24 - 11p - 11q + 4pq}.$$

$p = 3$ and $q = h^2 + 9$ for some integer $h$

$g = h^2 + h + 9$
Lemma 2. For integers, $k, p, q \geq 1$, let $G = \Phi(k, p, q)$ with girth $g = 2k + 1$. Then,

$$W(L(G)) - W(G) = \frac{1}{2} \left( g^2 + (p - q)^2 + 5(p + q - 3) - 2g(p + q - 3) \right).$$

$$g = -3 + p + q + \sqrt{24 - 11p - 11q + 4pq}.$$ 

$p = 3$ and $q = h^2 + 9$ for some integer $h$ 

$g = h^2 + h + 9$ 

By Claim 1, for $a_0 = 3$ and $a_1 = h^2 + 9$
Lemma 2. For integers, $k, p, q \geq 1$, let $G = \Phi(k, p, q)$ with girth $g = 2k + 1$. Then,

$$W(L(G)) - W(G) = \frac{1}{2} \left( g^2 + (p - q)^2 + 5(p + q - 3) - 2g(p + q - 3) \right).$$

$$g = -3 + p + q + \sqrt{24 - 11p - 11q + 4pq}.$$

$p = 3$ and $q = h^2 + 9$ for some integer $h$

$g = h^2 + h + 9$

By Claim 1, for $a_0 = 3$ and $a_1 = h^2 + 9$

an infinite family of graphs $G$ satisfying $W(L(G)) = W(G)$
Families of integer solutions

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$g$</th>
<th>$24 - 11p - 11q + 4pq$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$h^2 + 9$</td>
<td>$h^2 + h + 9$</td>
<td>$h^2$</td>
</tr>
<tr>
<td>4</td>
<td>$20h^2 + 4$</td>
<td>$20h^2 + 10h + 5$</td>
<td>$(10h)^2$</td>
</tr>
<tr>
<td>6</td>
<td>$13h^2 + 12h + 6$</td>
<td>$13h^2 + 25h + 15$</td>
<td>$(13h + 6)^2$</td>
</tr>
<tr>
<td>6</td>
<td>$13h^2 + 14h + 7$</td>
<td>$13h^2 + 27h + 17$</td>
<td>$(13h + 7)^2$</td>
</tr>
<tr>
<td>7</td>
<td>$17h^2 + 14h + 6$</td>
<td>$17h^2 + 31h + 17$</td>
<td>$(17h + 7)^2$</td>
</tr>
<tr>
<td>7</td>
<td>$17h^2 + 20h + 9$</td>
<td>$17h^2 + 37h + 23$</td>
<td>$(17h + 10)^2$</td>
</tr>
<tr>
<td>9</td>
<td>$h^2 + 3$</td>
<td>$h^2 + 5h + 9$</td>
<td>$(5h)^2$</td>
</tr>
<tr>
<td>10</td>
<td>$29h^2 + 2h + 3$</td>
<td>$29h^2 + 31h + 11$</td>
<td>$(29h + 1)^2$</td>
</tr>
<tr>
<td>10</td>
<td>$29h^2 + 56h + 30$</td>
<td>$29h^2 + 85h + 65$</td>
<td>$(29h + 28)^2$</td>
</tr>
<tr>
<td>12</td>
<td>$37h^2 + 30h + 9$</td>
<td>$37h^2 + 67h + 31$</td>
<td>$(37h + 15)^2$</td>
</tr>
<tr>
<td>12</td>
<td>$37h^2 + 44h + 16$</td>
<td>$37h^2 + 81h + 45$</td>
<td>$(37h + 22)^2$</td>
</tr>
<tr>
<td>13</td>
<td>$41h^2 + 4h + 3$</td>
<td>$41h^2 + 45h + 12$</td>
<td>$(41h + 2)^2$</td>
</tr>
<tr>
<td>13</td>
<td>$41h^2 + 78h + 40$</td>
<td>$41h^2 + 119h + 86$</td>
<td>$(41h + 39)^2$</td>
</tr>
<tr>
<td>16</td>
<td>$53h^2 + 44h + 12$</td>
<td>$53h^2 + 97h + 41$</td>
<td>$(53h + 22)^2$</td>
</tr>
<tr>
<td>16</td>
<td>$53h^2 + 62h + 21$</td>
<td>$53h^2 + 115h + 59$</td>
<td>$(53h + 31)^2$</td>
</tr>
<tr>
<td>18</td>
<td>$61h^2 + 116h + 58$</td>
<td>$61h^2 + 177h + 123$</td>
<td>$(61h + 58)^2$</td>
</tr>
<tr>
<td>18</td>
<td>$61h^2 + 128h + 70$</td>
<td>$61h^2 + 189h + 141$</td>
<td>$(61h + 64)^2$</td>
</tr>
</tbody>
</table>
Bounds on Gutman Index
Definition

\( G = (V, E) \) a finite, simple and undirected graph
Definition

\[ G = (V, E) \] a finite, simple and undirected graph

Gutman index

\[
\sum_{u,v \in V(G)} d(u)d(v)d(u, v)
\]
Definition

$G = (V, E)$ a finite, simple and undirected graph

Gutman index

$$\sum_{u, v \in V(G)} d(u)d(v)d(u, v)$$

Schultz index of the second kind
Related results
Related results

**Theorem 8** (Dankelmann, Gutman, Mukwembi, Swart, 2009). Let $G$ be a connected graph on $n$ vertices. Then

$$\text{Gut}(G) \leq \frac{2^4}{5^5} n^5 + O \left( n^{9/2} \right),$$

and the coefficient of $n^5$ is the best possible.
Related results

**Theorem 8** (Dankelmann, Gutman, Mukwembi, Swart, 2009). Let $G$ be a connected graph on $n$ vertices. Then

$$\text{Gut}(G) \leq \frac{2^4}{5^5} n^5 + O \left( n^{\frac{9}{2}} \right),$$

and the coefficient of $n^5$ is the best possible.

**Theorem 9** (Gutman, 2004). Let $T$ be a tree on $n$ vertices. Then

$$\text{Gut}(T) = 4W(T) - (2n - 1)(n - 1).$$
The graph with minimal Gutman index

**Theorem 10** (Andova, D., Fink, Škrekovski, 2011). *For every graph $G$ on $n$ vertices, it holds that*

$$(2n - 3)(n - 1) = \text{Gut}(S_n) \leq \text{Gut}(G).$$

*The equality holds if and only if $G$ is star $S_n$.***
Proof.

\[ \delta(G) \geq 2 \]
Proof.

\[ \delta(G) \geq 2 \]

\[ \text{Gut}(G) = \sum_{u,v \in V(G)} d(u)d(v)d(u,v) \]
Proof.

\[ \delta(G) \geq 2 \]

\[
\text{Gut}(G) = \sum_{u,v \in V(G)} d(u)d(v)d(u,v) \geq 4 \sum_{u,v \in V(G)} d(u,v)
\]
Proof.

\[ \delta(G) \geq 2 \]

\[ \text{Gut}(G) = \sum_{u,v \in V(G)} d(u)d(v)d(u,v) \geq 4 \sum_{u,v \in V(G)} d(u,v) \geq 4 \binom{n}{2} \]
Proof.

\[ \delta(G) \geq 2 \]

\[ \text{Gut}(G) = \sum_{u,v \in V(G)} d(u)d(v)d(u,v) \geq 4 \sum_{u,v \in V(G)} d(u,v) \]

\[ \geq 4 \binom{n}{2} = 2n(n-1) > (2n-3)(n-1) \]
Proof.

\[ \delta(G) \geq 2 \]

\[
\begin{align*}
\text{Gut}(G) &= \sum_{u,v \in V(G)} d(u)d(v)d(u,v) \\
&\geq 4 \sum_{u,v \in V(G)} d(u,v) \\
&\geq 4 \binom{n}{2} \\
&= 2n(n-1) > (2n-3)(n-1) = \text{Gut}(S_n)
\end{align*}
\]
\( \delta(G) = 1 \)
$\delta(G) = 1$ 

induction on the number of vertices
\[ \delta(G) = 1 \]

induction on the number of vertices

n=1
\[ \delta(G) = 1 \]

induction on the number of vertices

\[ n=1 \quad G \text{ on } n \text{ vertices} \]
\( \delta(G) = 1 \)

induction on the number of vertices

\( n=1 \quad G \text{ on } n \text{ vertices} \)

\( G' \text{ on } n + 1 \text{ vertices} \)
\( \delta(G') = 1 \)

induction on the number of vertices

\( n=1 \quad \text{on } n \text{ vertices} \)

\( G' \) on \( n + 1 \) vertices
\[ \delta(G) = 1 \]

\[ n=1 \quad G \text{ on } n \text{ vertices} \]

\[ G' \text{ on } n + 1 \text{ vertices} \]

induction on the number of vertices
\[ \delta(G) = 1 \]

Induction on the number of vertices

\[ n=1 \quad G \text{ on } n \text{ vertices} \]

\[ G' \text{ on } n + 1 \text{ vertices} \]

\[ D_G(u, v) = d_G(u)d_G(v)d_G(u, v) \]
\[ \delta(G) = 1 \]

\[ \text{induction on the number of vertices} \]

\[ n=1 \quad G \text{ on } n \text{ vertices} \]

\[ G' \text{ on } n + 1 \text{ vertices} \]

\[ D_G(u,v) = d_G(u)d_G(v)d_G(u,v) \]

\[ \text{Gut}(G') - \text{Gut}(G) = \sum_{v \in V(G) \setminus \{a\}} [D_{G'}(a,v) - D_G(a,v)] + \sum_{v \in V(G)} D_{G'}(x,v) \]

\[ + \sum_{u,v \in V(G) \setminus \{a\}} [D_{G'}(u,v) - D_G(u,v)] \]
\[ \delta(G) = 1 \]

\[ n = 1 \] 

\( G \) on \( n \) vertices

\( G' \) on \( n + 1 \) vertices

\[ D_G(u, v) = d_G(u)d_G(v)d_G(u, v) \]

\[ \text{induction on the number of vertices} \]

\[ \text{Gut}(G') - \text{Gut}(G) = \sum_{v \in V(G) \setminus \{a\}} [D_{G'}(a, v) - D_G(a, v)] + \sum_{v \in V(G)} D_{G'}(x, v) \]

\[ + \sum_{u, v \in V(G) \setminus \{a\}} [D_{G'}(u, v) - D_G(u, v)] \]

\[ d_{G'}(u, v) = d_G(u, v) \]
\[ \delta(G) = 1 \]

induction on the number of vertices

\[ n=1 \quad G \text{ on } n \text{ vertices} \]

\[ G' \text{ on } n+1 \text{ vertices} \]

\[ D_G(u, v) = d_G(u)d_G(v)d_G(u, v) \]

\[ \text{Gut}(G') - \text{Gut}(G) = \sum_{v \in V(G) \setminus \{a\}} [D_{G'}(a, v) - D_G(a, v)] + \sum_{v \in V(G)} D_{G'}(x, v) + \sum_{u, v \in V(G) \setminus \{a\}} [D_{G'}(u, v) - D_G(u, v)] \]

\[ d_{G'}(u, v) = d_G(u, v) \quad d_{G'}(x, v) = 1 + d_G(a, v) \text{ for every } u, v \in V(G) \]
\( \delta(G) = 1 \)  

induction on the number of vertices

\( n=1 \quad G \) on \( n \) vertices

\( G' \) on \( n + 1 \) vertices

\[ D_G(u, v) = d_G(u)d_G(v)d_G(u, v) \]

\[ \text{Gut}(G') - \text{Gut}(G) = \sum_{v \in V(G) \setminus \{a\}} [D_{G'}(a, v) - D_G(a, v)] + \sum_{v \in V(G)} D_{G'}(x, v) \]

\[ + \sum_{u, v \in V(G) \setminus \{a\}} [D_{G'}(u, v) - D_G(u, v)] \]

\[ d_{G'}(u, v) = d_G(u, v) \quad d_{G'}(x, v) = 1 + d_G(a, v) \text{ for every } u, v \in V(G) \]

only the degree of \( a \) increases
Gut\((G') - Gut(G) = \sum_{v \in V(G) \setminus \{a\}} d_G(v)d_G(a, v) + \sum_{v \in V(G)} d_{G'}(v)(d_G(a, v) + 1) = 2 \sum_{v \in V(G) \setminus \{a\}} d_G(v)d_G(a, v) + \sum_{v \in V(G)} d_G(v) + 1
\[ \text{Gut}(G') - \text{Gut}(G) = \sum_{v \in V(G) \setminus \{a\}} d_G(v)d_G(a, v) + \sum_{v \in V(G)} d_{G'}(v)(d_G(a, v) + 1) \]

\[ = 2 \sum_{v \in V(G) \setminus \{a\}} d_G(v)d_G(a, v) + \sum_{v \in V(G)} d_G(v) + 1 \]

\[ \sum_{v \in V(G)} d_G(v) = 2|E(G)| \geq 2(n - 1) \]
\[ \text{Gut}(G') - \text{Gut}(G) = \sum_{v \in V(G) \setminus \{a\}} d_G(v)d_G(a, v) + \sum_{v \in V(G)} d_{G'}(v)(d_G(a, v) + 1) \]

\[ = 2 \sum_{v \in V(G) \setminus \{a\}} d_G(v)d_G(a, v) + \sum_{v \in V(G)} d_G(v) + 1 \]

\[ \sum_{v \in V(G)} d_G(v) = 2|E(G)| \geq 2(n - 1) \]

\[ \text{Gut}(G') - \text{Gut}(G) \geq 4n - 3 \]
\[
\text{Gut}(G') - \text{Gut}(G) = \sum_{v \in V(G) \setminus \{a\}} d_G(v)d_G(a, v) + \sum_{v \in V(G)} d_{G'}(v)(d_G(a, v) + 1)
\]

\[
= 2 \sum_{v \in V(G) \setminus \{a\}} d_G(v)d_G(a, v) + \sum_{v \in V(G)} d_G(v) + 1
\]

\[
\sum_{v \in V(G)} d_G(v) = 2|E(G)| \geq 2(n - 1)
\]

\[
\text{Gut}(G') - \text{Gut}(G) \geq 4n - 3 = \text{Gut}(S_{n+1}) - \text{Gut}(S_n)
\]
Theorem 11 (Gutman, 2003).

\[
\binom{n}{2} = W(K_n) \leq W(G) \leq W(P_n) = \binom{n+1}{3}
\]
Theorem 11 (Gutman, 2003).

\[ \binom{n}{2} = W(K_n) \leq W(G) \leq W(P_n) = \binom{n+1}{3} \]

Theorem 12 (Gutman, 2004). Let $T$ be a tree on $n$ vertices. Then

\[
\text{Gut}(T) = 4W(T) - (2n - 1)(n - 1)
\]
**Theorem 11** (Gutman, 2003).

\[
\binom{n}{2} = W(K_n) \leq W(G) \leq W(P_n) = \binom{n+1}{3}
\]

**Theorem 12** (Gutman, 2004). Let \( T \) be a tree on \( n \) vertices. Then

\[
\text{Gut}(T) = 4W(T) - (2n - 1)(n - 1)
\]

**Corollary 1.** For every tree \( T \) on \( n \) vertices, it holds that

\[
(n-1)(2n-3) = \text{Gut}(S_n) \leq \text{Gut}(T) \leq \text{Gut}(P_n) = \frac{(n-1)(2n^2 - 4n + 3)}{3}
\]
Bounds on graphs with minimal Gutman index, $\delta \geq 2$

**Lemma 3.** A connected graph $G$ on $n$ vertices with minimum degree at least $\delta \geq 2$ and minimal Gutman index satisfies

$$\delta(\delta + 1)n^2 > \text{Gut}(G) \geq \frac{\delta^2 n}{2} (2n - \delta - 2).$$
Bounds on graphs with minimal Gutman index, $\delta \geq 2$

**Lemma 3.** A connected graph $G$ on $n$ vertices with minimum degree at least $\delta \geq 2$ and minimal Gutman index satisfies

$$\delta(\delta + 1)n^2 > \text{Gut}(G) \geq \frac{\delta^2 n}{2}(2n - \delta - 2).$$
Gut\((G) \geq \frac{\delta^2 n}{2} (2n - \delta - 2)\)
\[ \text{Gut}(G) \geq \frac{\delta^2 n}{2} (2n - \delta - 2) \]

\[ \text{Gut}(G) = \sum_{u,v} d(v) d(u, v) = \frac{1}{2} \sum_u d(u) \sum_v d(v) d(u, v) \]
\[ \text{Gut}(G) \geq \frac{\delta^2 n}{2} (2n - \delta - 2) \]

\[
\text{Gut}(G) = \sum_{u,v} d(v) d(u, v) = \frac{1}{2} \sum_u d(u) \sum_v d(v) d(u, v)
\]
\[
\geq \frac{n}{2} \min_u \left( d(u) \sum_v d(v) d(u, v) \right)
\]
\[ \text{Gut}(G) \geq \frac{\delta^2 n}{2} (2n - \delta - 2) \]

\[
\text{Gut}(G) = \sum_{u,v} d(v) d(u, v) = \frac{1}{2} \sum_u d(u) \sum_v d(v) d(u, v) \\
\geq \frac{n}{2} \min_u \left( d(u) \sum_v d(v) d(u, v) \right) \\
\geq \frac{n \delta}{2} \min_u \left( d(u) \sum_v d(u, v) \right)
\]
Gut(G) \geq \frac{\delta^2 n}{2} (2n - \delta - 2)

Gut(G) = \sum_{u,v} d(v) d(u, v) = \frac{1}{2} \sum_u d(u) \sum_v d(v) d(u, v)

\geq \frac{n}{2} \min_u \left( \sum_v d(v) d(u, v) \right)

\geq \frac{n \delta}{2} \min_u \left( \sum_v d(u, v) \right)

d(u) \text{ vertices on distance one to } u
Gut(G) ≥ \( \delta^2 \frac{n}{2} (2n - \delta - 2) \)

\[
Gut(G) = \sum_{u,v} d(v) d(u, v) = \frac{1}{2} \sum_u d(u) \sum_v d(v) d(u, v)
\]

\[
\geq \frac{n}{2} \min_u \left( d(u) \sum_v d(v) d(u, v) \right)
\]

\[
\geq \frac{n \delta}{2} \min_u \left( d(u) \sum_v d(u, v) \right)
\]

\(d(u)\) vertices on distance one to \(u\)

\(n - d(u) - 1\) vertices on distance at least two to \(u\)

\[
Gut(G) \geq \frac{n \delta}{2} \min_u \left( d(u)(2n - d(u) - 2) \right)
\]
\( \text{Gut}(G) \geq \frac{\delta^2 n}{2} (2n - \delta - 2) \)

\[
\text{Gut}(G) = \sum_{u,v} d(v) d(u,v) = \frac{1}{2} \sum_u d(u) \sum_v d(v) d(u,v)
\]

\[
\geq \frac{n}{2} \min_u \left( d(u) \sum_v d(v) d(u,v) \right)
\]

\[
\geq \frac{n \delta}{2} \min_u \left( d(u) \sum_v d(u,v) \right)
\]

d\( (u) \) vertices on distance one to \( u \)

\( n - d(u) - 1 \) vertices on distance at least two to \( u \)

\[
\text{Gut}(G) \geq \frac{n \delta}{2} \min_u \left( d(u)(2n - d(u) - 2) \right)
\]

\( d(u)(2n - d(u) - 2) \) with \( \delta \leq x \leq n - 1 \) has its minimum at \( \delta \)
\[ \text{Gut}(G) \geq \frac{\delta^2 n}{2} (2n - \delta - 2) \]

\[ \text{Gut}(G) = \sum_{u,v} d(v) d(u,v) = \frac{1}{2} \sum_u d(u) \sum_v d(v) d(u,v) \]

\[ \geq \frac{n}{2} \min_u \left( d(u) \sum_v d(v) d(u,v) \right) \]

\[ \geq \frac{n \delta}{2} \min_u \left( d(u) \sum_v d(u,v) \right) \]

d\(u\) vertices on distance one to \(u\)

\(n - d(u) - 1\) vertices on distance at least two to \(u\)

\[ \text{Gut}(G) \geq \frac{n \delta}{2} \min_u \left( d(u)(2n - d(u) - 2) \right) \]

d\(u\)(2\(n - d(u) - 2\)) with \(\delta \leq x \leq n - 1\) has its minimum at \(\delta\)

\[ \text{Gut}(G) \geq \frac{\delta^2 n}{2} (2n - \delta - 2) \]
\( \delta (\delta + 1)n^2 > \text{Gut}(G) \)
\[ \delta(\delta + 1)n^2 > \text{Gut}(G) \]

a graph \( H \) on \( n - 1 \) vertices
\( \delta(\delta + 1)n^2 > \text{Gut}(G) \)

a graph \( H \) on \( n - 1 \) vertices
\[ \delta(\delta + 1)n^2 > \text{Gut}(G) \]
\[ \delta(\delta + 1)n^2 > \text{Gut}(G) \]

A graph \( H^* \) on \( n \) vertices
\[ \delta(\delta + 1)n^2 > \text{Gut}(G) \]
\[ \delta(\delta + 1)n^2 > \text{Gut}(G) \]
\[ \delta(\delta + 1)n^2 > \text{Gut}(G) \]

The contribution of \( y \) to \( \text{Gut}(H^*) \) is

\[ \sum_{v \in V(H)} d_{H^*}(y)d_{H^*}(v)d_{H^*}(y,v) \leq (n - 1)((n - 2)\delta + \delta + 1). \]
\( \delta(\delta + 1)n^2 > \text{Gut}(G) \)

The contribution of \( y \) to \( \text{Gut}(H^*) \) is
\[
\sum_{v \in V(H)} d_{H^*}(y)d_{H^*}(v)d_{H^*}(y,v) \leq (n - 1)((n - 2)\delta + \delta + 1).
\]

The contribution of \( x \) to \( \text{Gut}(H^*) \) is
\[
\sum_{v \in V(H)} d_{H^*}(x)d_{H^*}(v)d_{H^*}(x,v) \leq (\delta + 1)(\delta^2 + 2\delta(n - \delta - 2)).
\]
\( \delta(\delta + 1)n^2 > \text{Gut}(G) \)

The contribution of \( y \) to \( \text{Gut}(H^*) \) is
\[
\sum_{v \in V(H)} d_{H^*}(y) d_{H^*}(v) d_{H^*}(y,v) \leq (n - 1)((n - 2)\delta + \delta + 1).
\]

The contribution of \( x \) to \( \text{Gut}(H^*) \) is
\[
\sum_{v \in V(H)} d_{H^*}(x) d_{H^*}(v) d_{H^*}(x,v) \leq (\delta + 1)(\delta^2 + 2\delta(n - \delta - 2)).
\]

the remaining vertices of \( H^* \) contribute with
\[
\sum_{v \in V(H)} d_{H^*}(x) d_{H^*}(v) d_{H^*}(x,v) \leq (\delta + 1)(\delta^2 + 2\delta(n - \delta - 2)).
\]
Corollary 2. A connected graph $G$ on $n$ vertices with minimum degree at least $\delta \geq 2$ and minimal Gutman index satisfies
\[
\delta(\delta + 1)n^2 - O(n) \geq \text{Gut}(G) \geq \delta^2 n^2 - O(n).
\]
Bounds on graphs with minimal Gutman index, $\Delta > 2$

**Lemma 4.** A connected graph $G$ on $n$ vertices with maximum degree at most $\Delta > 2$ and minimal Gutman index satisfies

$$\text{Gut}(G) < 4(n^2 - 8n + 4) \log_{\Delta-1} n.$$
Bounds on graphs with minimal Gutman index, $\Delta > 2$

**Lemma 4.** A connected graph $G$ on $n$ vertices with maximum degree at most $\Delta > 2$ and minimal Gutman index satisfies

$$\text{Gut}(G) < 4(n^2 - 8n + 4) \log_{\Delta-1} n.$$ 

a $\Delta$-regular balanced tree on $n$ vertices
A leave and an inner vertex: since every inner vertex has degree $\Delta$, and the distance between any two vertices is at most $2k$, these pairs contribution in the sum is less than

$$\frac{(\Delta - 2)n + 2}{\Delta - 1} \cdot \frac{n - 2}{\Delta - 1} 2k\Delta$$
A leave and an inner vertex: since every inner vertex has degree $\Delta$, and the distance between any two vertices is at most $2k$, these pairs contribution in the sum is less than

$$\frac{(\Delta - 2)n + 2}{\Delta - 1} \cdot \frac{n - 2}{\Delta - 1} \cdot 2k \Delta$$

Two leaves: the distance between two leaves is at most $\text{diam}(G) = 2k$, so their contribution to the Gutman index is at most

$$\left(\frac{\frac{(\Delta - 2)n + 2}{\Delta - 1}}{2}\right)^{2k} < \left(\frac{(\Delta - 2)n + 2}{\Delta - 1}\right)^k$$
A leave and an inner vertex: since every inner vertex has degree $\Delta$, and the distance between any two vertices is at most $2k$, these pairs contribution in the sum is less than

$$\frac{(\Delta - 2)n + 2}{\Delta - 1} \cdot \frac{n - 2}{\Delta - 1} 2k\Delta$$

Two leaves: the distance between two leaves is at most $\text{diam}(G) = 2k$, so their contribution to the Gutman index is at most

$$\left( \frac{(\Delta - 2)n + 2}{\Delta - 1} \right)^2 2k < \left( \frac{(\Delta - 2)n + 2}{\Delta - 1} \right)^k$$

Two inner vertices: these pair contribute at most

$$\left( \frac{n - 2}{\Delta - 1} \right)^2 2k\Delta^2 < \left( \frac{n - 2}{\Delta - 1} \right)^2 \Delta^2 k$$
A leave and an inner vertex: since every inner vertex has degree $\Delta$, and the distance between any two vertices is at most $2k$, these pairs contribution in the sum is less than
\[
\frac{(\Delta - 2)n + 2}{\Delta - 1} \cdot \frac{n - 2}{\Delta - 1} 2k\Delta
\]

Two leaves: the distance between two leaves is at most $\text{diam}(G) = 2k$, so their contribution to the Gutman index is at most
\[
\left( \frac{(\Delta - 2)n + 2}{\Delta - 1} \right) 2k < \left( \frac{(\Delta - 2)n + 2}{\Delta - 1} \right)^2 k
\]

Two inner vertices: these pair contribute at most
\[
\left( \frac{n - 2}{\Delta - 1} \right) 2k\Delta^2 < \left( \frac{n - 2}{\Delta - 1} \right)^2 \Delta^2 k
\]

$\text{Gut}(G) < 4(n^2 - 8n + 4) \log_{\Delta - 1} n$
Bounds on graphs with maximal Gutman index,
with maximum degree $\Delta(G) \leq \Delta$

**Lemma 5.** Let $G$ be a connected graph on $n$ vertices with maximum degree $\Delta(G) \leq \Delta$, and maximal Gutman index. Then, the following holds:

$$\frac{(n + 1)^3}{27} \Delta^2 \leq \text{Gut}(G) \leq \binom{n + 1}{3} \Delta^2.$$
\[ \frac{(n+1)^3}{27} \Delta^2 \leq \text{Gut}(G) \]
A pair \((x, y)\), where \(x\) is a vertex from \(Q^L\) and \(y\) is a vertex from \(Q^R\), contributes to \(\text{Gut}(Q)\) at least \(\Delta^2(bn + 1)\)
\[ \frac{(n+1)^3}{27} \Delta^2 \leq \text{Gut}(G) \]

A pair \((x, y)\), where \(x\) is a vertex from \(Q^L\) and \(y\) is a vertex from \(Q^R\), contributes to \(\text{Gut}(Q)\) at least \(\Delta^2 (bn + 1)\).

Since there are \(an\) vertices in bough, \(Q^L\) and \(Q^R\), the contribution of these vertices is \((an)^2 \Delta^2 (bn + 1)\).
\[
\frac{(n+1)^3}{27} \Delta^2 \leq \text{Gut}(G)
\]

A pair \((x, y)\), where \(x\) is a vertex from \(Q^L\) and \(y\) is a vertex from \(Q^R\), contributes to \(\text{Gut}(Q)\) at least \(\Delta^2(bn + 1)\)

Since there are \(an\) vertices in bough, \(Q^L\) and \(Q^R\), the contribution of these vertices is \((an)^2 \Delta^2(bn + 1)\)

constraint \(2a + b = 1\)
\[ \frac{(n+1)^3}{27} \Delta^2 \leq \text{Gut}(G) \]

A pair \((x, y)\), where \(x\) is a vertex from \(Q^L\) and \(y\) is a vertex from \(Q^R\), contributes to \(\text{Gut}(Q)\) at least \(\Delta^2 (bn + 1)\).

Since there are \(an\) vertices in bough, \(Q^L\) and \(Q^R\), the contribution of these vertices is \((an)^2 \Delta^2 (bn + 1)\).

The constraint \(2a + b = 1\)

\((an)^2 \Delta^2 (bn + 1)\) attains maximum for \(a = (n+1)/3n\) and \(b = \frac{n-2}{3n}\).
Gut(G') \leq \binom{n+1}{3} \Delta^2
\[ \text{Gut}(G) \leq \binom{n+1}{3} \Delta^2 \]

\[ \text{Gut}(G) \leq \sum_{u,v} \Delta^2 d(u, v) \]
Gut(G) \leq \binom{n+1}{3} \Delta^2

Gut(G) \leq \sum_{u,v} \Delta^2 d(u,v) = \Delta^2 W(G)
Gut(G) \leq \binom{n+1}{3} \Delta^2

Gut(G) \leq \sum_{u,v} \Delta^2 d(u,v) = \Delta^2 W(G) \leq \Delta^2 W(P_n)
\[ \text{Gut}(G) \leq \binom{n+1}{3} \Delta^2 \]

\[ \text{Gut}(G) \leq \sum_{u,v} \Delta^2 d(u, v) = \Delta^2 W(G) \leq \Delta^2 W(P_n) = \Delta^2 \left( \binom{n+1}{3} \right) \]
Corollary 3. Let $G$ be a graph on $n$ vertices with bounded maximum degree $\Delta$. Then,

$$O(n^3) \geq \text{Gut}(G) \geq \Omega(n^2 \log n),$$

and those bounds can be attained.
Bounds on graphs
with maximal Gutman index and minimum degree at least $\delta$
Bounds on graphs
with maximal Gutman index and minimum degree at least $\delta$

**Theorem 13** (Dankelmann, Gutman, Mukwembi, Swart, 2009). Let $G$ be a connected graph on $n$ vertices. Then

$$\text{Gut}(G) \leq \frac{2^4}{5^5} n^5 + O \left(n^{\frac{9}{2}}\right),$$

and the coefficient of $n^5$ is the best possible.
Bounds on graphs
with maximal Gutman index and minimum degree at least \( \delta \)

**Lemma 6.** A connected graph \( G \) on \( n \) vertices with minimum degree at least \( \delta \), and maximal Gutman index satisfies

\[
\frac{2^5}{5^5} \frac{(n + \delta - 1)^5}{\delta^5} < \text{Gut}(G).
\]
$$\frac{2^5}{5^5} \frac{(n+\delta-1)^5}{\delta^5} < \text{Gut}(G)$$
\[ \frac{2^5}{5^5} \frac{(n+\delta-1)^5}{\delta^5} < \text{Gut}(G) \]

\[ L \text{ has } n \text{ vertices, } 2an + bn\delta + 1 = n \]
\[
\frac{2^5}{5^5} \frac{(n+\delta-1)^5}{\delta^5} < \text{Gut}(G)
\]

\[L \text{ has } n \text{ vertices, } 2an + bn\delta + 1 = n\]

the contribution of the pairs \((x, y)\) to \text{Gut}(L), where \(x \in V(K^1_{an}), y \in V(K^2_{an})\),
which is more than \((an)^4 2(bn + 1)\)
\[
\frac{2^5}{5^5} \left( \frac{n+\delta-1}{\delta} \right)^5 < \text{Gut}(G)
\]

\[L\] has \(n\) vertices, \(2an + bn\delta + 1 = n\)

the contribution of the pairs \((x, y)\) to \(\text{Gut}(L)\), where \(x \in V(K^1_{an}), y \in V(K^2_{an})\), which is more than \((an)^4 2(bn + 1)\)

\(2(an)^4 (bn + 1)\) attains the maximum at \(bn + 1 = \frac{n+\delta-1}{5\delta}\) and \(an = 2(bn + 1)\)