Finding the k most vital nodes in an interval graph

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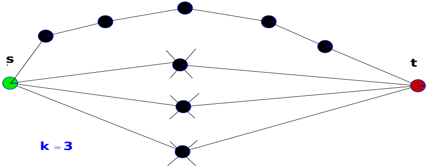
k most vital nodes

Given an undirected graph G = (V,E), For two specified nodes s and t in V, the k most vital nodes are those k nodes whose removal maximizes the increase in the length of the shortest path



from s to t.

Previous Work

Defined by
W.H. Corley and D.Y. Sha, "Most vital links and nodes in weighted networks", Operations Research Letters, 1:157-160, September 1982.

Shown to be NP-complete by
A. Bar-Noy, S. Khuller and B. Schieber, "The complexity of Finding Most Vital Arcs and Nodes", Technical Report CS-TR-3539, University of Maryland, 1995.

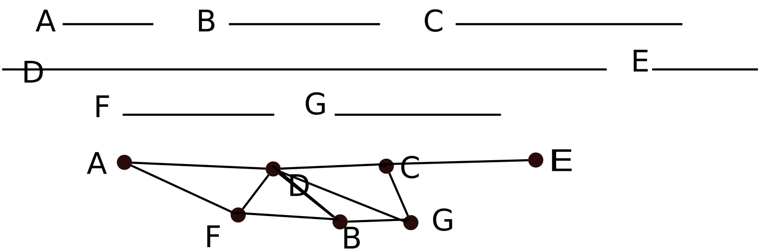
Related inapproximability bounds

If 0-1 lengths are placed on the edges and l_v is the maximum of the s-t distances in all graphs obtained from G by removing k nodes then it is NP-hard to approximate l_v to within a factor smaller than 2, even for bipartite graphs.

L. Khachiyan, E. Boros, K. Borys, K. Elbassioni, V. Gurvich, G. Rudolf and J. Zhao, "On Short Path Interdiction Problems: Total and Node-Wise Limited Interdiction", Theory Computer Systems 43 (2008), 204-233.

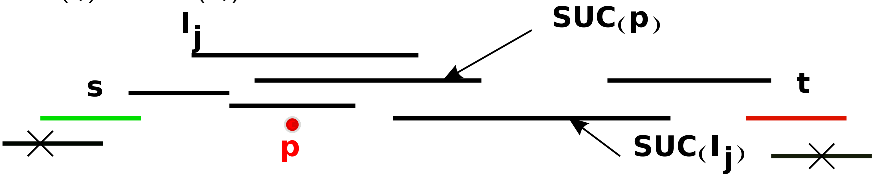
Interval Graphs

Each interval graph has an intersection model where each vertex corresponds to an interval of the real line and two vertices are adjacent iff their corresponding intervals have a non-empty intersection.



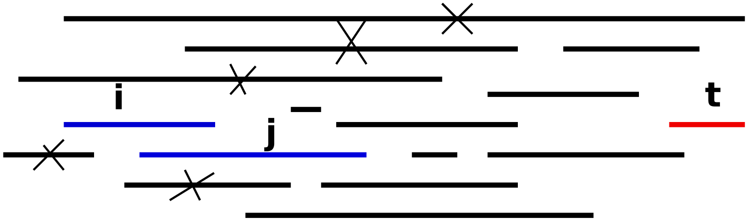
Successors

Wlog assume t is right of s and that s and t are not adjacent. Order the intervals by increasing right endpoint. Wlog assume s = $[a_1, b_1]$ as intervals left of s do not contribute to a shortest s–t path in any subgraph of G. Likewise assume that intervals I_j s.t. $a_j \geq a_t$ have been pruned. For every point p on the real line define the sucessor function as follows:
$SUC(p) = I_i \in I \iff b_i = max\{b_j | I_j \text{ contains p }\}$ Also $SUC(I_i) = SUC(b_i)$.



Subproblems

Define $H_{i,j}(j > i)$ to be the induced subgraph of G consisting of vertex i, vertex j adjacent to i, and all vertices of G whose corresponding intervals have their left endpoints right of b_i. Note that in $H_{i,j}$, vertex j is $SUC(i)$.



Dynamic Programming

Define $M(i, j, u)$ to be the length of the longest shortest path from interval I_i to t in an induced subgraph of $H_{i,j}$ after u vertices have been deleted. (i and j cannot be deleted)

The overall problem for G is to find $Max_{j \text{ adjacent to } s}\{M(s, j, v)\}$ where j is the interval with the $(q + 1)$st rightmost endpoint of the intervals adjacent to s in G, and $v = k - q$.

Computation

To compute $M(i, j, u)$

If $u = 0$ then $M(i, j, u)$ is equal to the length of the shortest i–t path in $H_{i,j}$
If i is adjacent to t then $M(i, j, u) = 1$ else if j is adjacent to t then $M(i, j, u) = 2$

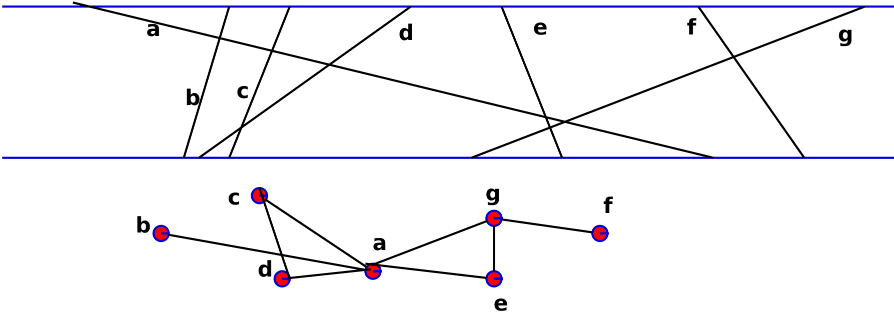
Otherwise, for $0 \leq q \leq u$ (Here the rightmost q remaining neighbours of j are deleted)
Let z be the $(q + 1)$st rightmost interval adjacent to j in $H_{i,j}$ then $M(i, j, u) = max_{0 \leq q \leq u}[1 + M(j, z, u - q)]$

Implementation and Timimg

We need only consider nk of the $H_{i,j}$ subgraphs as only the k rightmost neighbours of i need to be considered as j. An additional factor of kn will be needed to compute shortest paths and to add in previously computed solutions.
Altogether the algorithm requires $O(n^2k^2)$ time.

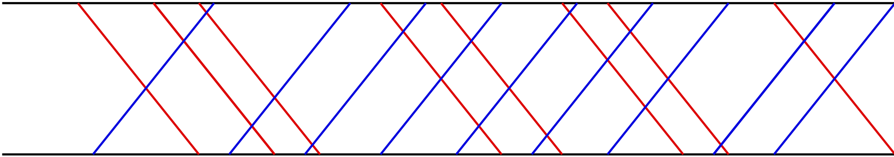
Permutation graphs

A graph is a permutation graph iff it the intersection graph of a set of segments connecting two horizontal lines



Theorem

A graph is a bipartite permutation graph iff it has an intersection model where all segments have the same length.



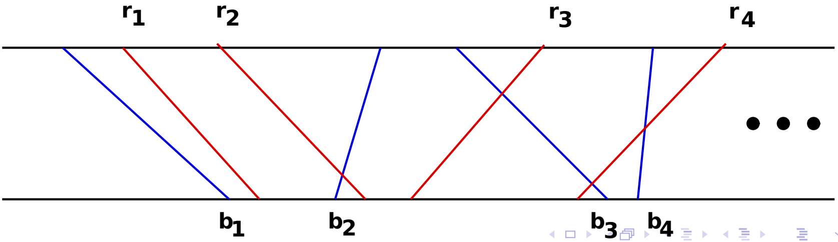
Proof:

Same length implies bipartite.

Given an intersection model I of a bipartite permutation graph

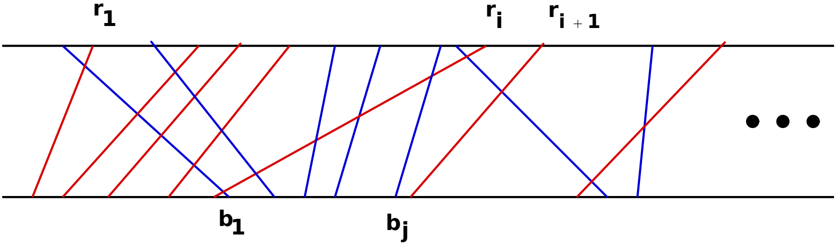
By induction

- Let $R = \{r_1, r_2, \ldots\}$ and $B = \{b_1, b_2, \ldots\}$ be the two color classes numbered from left to right
- Note that any red segment intersects a contiguous range of the blue segments in their natural order.
- If r_1 and b_1 do not cross, then then the leftmost segment of I is isolated, and by the inductive assumption the other segments can be placed and then then the leftmost positioned so that r_1 and b_1 do not cross.



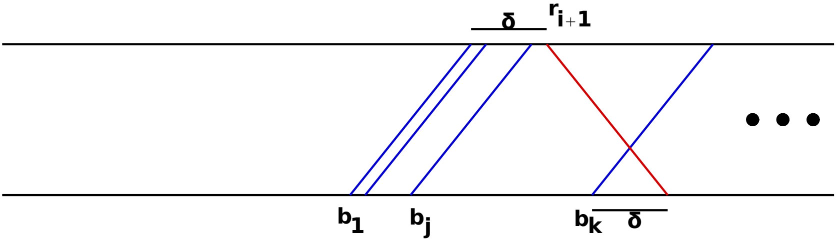
If r_1 and b_1 do cross then

Let $r_1, r_2, \ldots, r_i$ be the segments of R that cross b_1 and let $b_1, b_2, \ldots, b_j$ be the segments of B entirely left of r_{i+1}. $J = I \setminus (\{r_1, r_2, \ldots, r_i\} \cup \{b_1, b_2, \ldots b_j\})$. By the inductive hypothesis there exists an intersection model J' for J in which all segments have the same length.



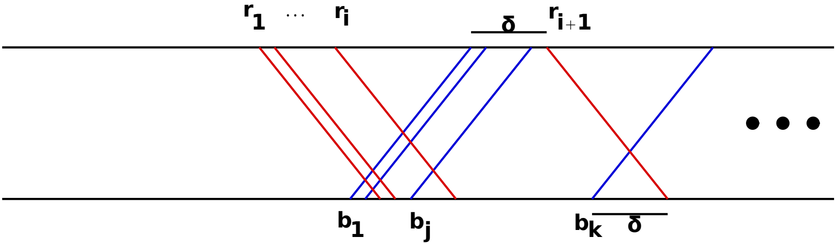
We will add $r_1, r_2, \ldots r_i$ and $b_1, b_2, \ldots b_j$ to J'.

Let b_k denote the rightmost neighbour of r_{i+1} in J' and δ the extent of the overlap between b_k and r_{i+1} in J'.

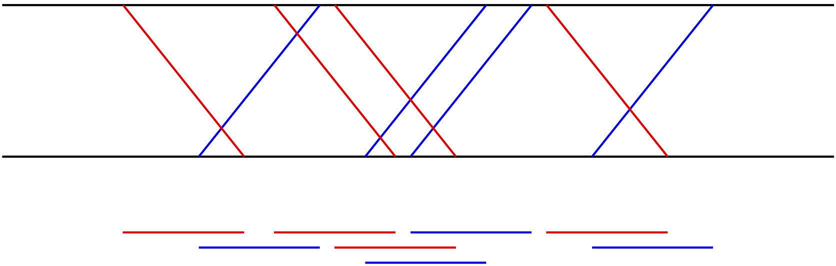


Place $b_1, b_2, \ldots b_j$ to the left of r_{i+1} but within δ of it.

Note that each interval in $\{r_1, r_2, \ldots r_i\}$ crosses a contiguous range of $b_1, b_2, \ldots b_k$ that includes b_1. Add the remaining segments



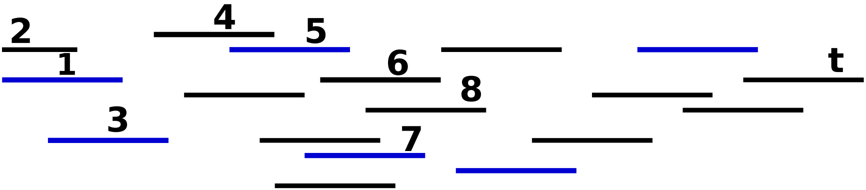
Bipartite permutation graphs



Project to x-axis to obtain colored unit interval representation

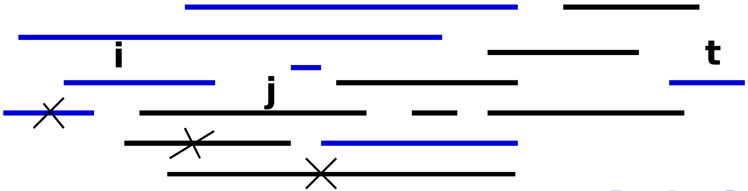
k most vital nodes problem

In the bipartitle graph the
shortest path can back up. Intervals of the same color must advance.



Subproblems

Define $H_{i,j}$ to be the induced subgraph of G consisting of vertex i, vertex j where j is adjacent to i and of opposite color, plus all vertices of the same color as i that extend to the right of i, plus all vertices of the same color as j that extend to the right of j and are not adjacent to i



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The overall problem for G is to find $Max_{j \text{ adjacent to } s \text{ and of opposite color}}\{M(s, j, v)\}$ where j is the interval with the $(q + 1)$st rightmost endpoint of the intervals adjacent to s in G, and $v = k - q$.

s and t are close

If s and t have opposite colors then
They are at distance 1 if and only if they are adjacent. They are at distance 3 if and only if they are not adjacent and the rightmost neighbour of opposite color of s is at distance 2 from t

If s and t have the same color then
They are at distance 2 if and only if there exits a vertex of opposite color to s and t yet adjacent to both.

Computation

To compute $M(i,j,u)$
If $u = 0$ then $M(i,j,u)$ is equal to the length of the shortest i –t path in $H_{i,j}$. Note that wlog we may assume that j lies on this path.
The case where t is close to s already been covered. Otherwise for $0 \leq q \leq u$. Let z be the $(q+1)$st rightmost interval of opposite color to j and adjacent to j in $H_{i,j}$ then
$M(i,j,u) = max_{0 \leq q \leq u}\{1 + M(j,z,u-q)\}$

Timing

There are $O(nk^2)$ table entries to be computed and each may require $O(k)$ time. In total the algorithm may require $O(nk^3)$ time.

Open Problems

It would be interesting to determine the complexity of the most vital nodes problem on other classes of graphs. Some classes that could be considered are:

chordal graphs

permutation graphs