On the dynamics of piezoactuated positioning systems

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Piezoelectric actuators (PEAs) are commercially available for producing extremely small displacements. If a PEA is used to drive a positioning mechanism, the combined dynamics is approximated empirically by that of a second-order system. However, the rationale for such an approximation is lacking in the literature, thus leaving some issues unaddressed, such as the scope of and the error associated with the approximation. This paper presents such a rationale for the second-order approximation and a method to quantify the error associated with the approximation, by employing the assumed mode method to solve the governing equation. For the experimental verification, step voltages of 48.2, 64.2, and 75.4 V were used to excite a prototype of the positioning system with a mass ratio of 24.7, 47.7, 87.5, 115.8, and 138.8, respectively; and the measured system step responses were compared to the ones simulated by using the second-order approximation. Also, it is illustrated that the error associated with the approximation can be well characterized and quantified by using the developed method.
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I. INTRODUCTION

Piezoactuated positioning systems have been found applications in many fields of technology and research, such as semiconductor manufacture and precision machining. In these applications, the behavior of the positioning system has proven to be of importance for its design and/or control. By considering the piezoelectric actuator (PEA) as a stand-alone system, Goldfarb and Celanovic[1] developed a model, based on the generalized Maxwell resistive capacitor, to represent the static hysteresis that is always observed in PEAs. Despite the presence of the nonlinear hysteresis, the dynamics of the PEA was observed to have simple second-order linear characteristics.[1] Adriaens *et al.*[2] elaborated on this model by including a first-order hysteresis effect and by modeling the PEA as a distributed parameter system. The authors concluded that if the positioning system was designed well, the aforementioned second-order approximation was a good approximation of the system dynamics. This empirical approximation has been also widely employed in the applications where the system dynamics is concerned, for example, the ones reported in Refs. 3 and 4. However, the rationale for such an approximation is lacking in the literature. This also leaves some issues unaddressed, such as the scope of and the error associated with the approximation. This note is to fill the void by providing readers with such a rationale and its experimental verification.

II. MODELING THE DYNAMICS OF PIEZOACTUATED POSITIONING SYSTEMS

Consider a PEA shown in Fig. 1, which is used to drive a positioning mechanism with a mass of M. Shown in Fig. 1 also includes an element with an infinitesimal thickness of dx. The forces on both sides of the element are $N(x,t)$ and $N(x+dx,t)$. Under the action of these forces, the element has moved a distance $\upsilon(x,t)$ from its equilibrium position at x, as measured from the fixed end of the PEA. Summing the forces on the element, one has

$$\rho A dx \frac{\partial \upsilon^2(x,t)}{\partial t^2} = N(x+dx,t) - N(x,t) = \frac{\partial N(x,t)}{\partial x} dx, \qquad (1)$$

where t is the time and ρ and A are the mass density and cross-sectional area of the PEA, respectively. Under the assumption that the PEA is viscoelastic, one can use the Kelvin–Voigt model to describe the relationship between the stress and the strain in the material, i.e.,

$$\sigma(x,t) = E\varepsilon(x,t) + \eta \frac{\partial \varepsilon(x,t)}{\partial t}, \qquad (2)$$

where $\sigma(x,t)$ and $\varepsilon(x,t)$ are the stress and strain and E and η are Young's modulus and the viscous coefficient of the PEA material, respectively. Using the definition of stress and strain, i.e., $\sigma(x,t)=N(x,t)/A$ and $\varepsilon(x,t)=\partial \upsilon(x,t)/\partial x$, the following equation can be obtained from Eqs. (1) and (2):

$$\rho \frac{\partial^2 \upsilon(x,t)}{\partial t^2} = E \frac{\partial^2 \upsilon(x,t)}{\partial x^2} + \eta \frac{\partial^3 \upsilon(x,t)}{\partial x^2 \partial t}. \qquad (3)$$

From Eq. (3), it is seen that $\upsilon(x,t)$ is governed by a partial differential equation. To seek a solution for such an equation, the boundary conditions need to be used. Based on the fact

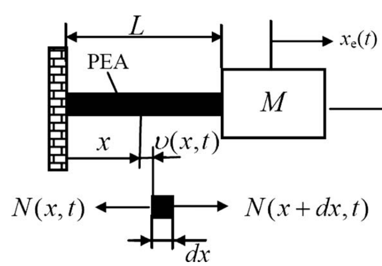

FIG. 1. Piezoactuated positioning system and a PEA infinitesimal element.

that the displacement at the fixed end of the PEA is zero and the fact that the force in the PEA at the other end equals to the inertial force of the positioning mechanism, the boundary conditions can be expressed mathematically by

$$\upsilon(0,t) = 0, \tag{4}$$

$$N(L,t) = -M\frac{\partial^2 \upsilon(x,t)}{\partial t^2}\bigg|_{x=L}. \tag{5}$$

By means of the assumed mode method, the solution to Eq. (3) is given by a linear combination of assumed modes with different frequencies, i.e.,

$$\upsilon(x,t) = \sum_{j=1}^{\infty} \phi_j(x) q_j(t), \tag{6}$$

where $\phi_j(x)$ is the normal mode function and $q_j(t)$ is the mode coordinate. Substituting Eq. (6) to Eq. (3) and utilizing the boundary conditions given in Eqs. (4) and (5), one has the following normal mode function:[5]

$$\phi_j(x) = B \sin(p_j x), \tag{7}$$

where B is a constant, depending on the initial condition of the system; and p_j is governed by the following equation:

$$p_j L \tan(p_j L) = \frac{m}{M}, \tag{8}$$

where $m=\rho AL$, which is the mass of the PEA. For given values of L, m, and M, one can solve for p from Eq. (8) numerically. It should be noted that p has an infinite number of values, corresponding to the orders of modes. If a voltage is applied to the PEA, by means of the aforementioned assumed modes, one has the mode coordinates governed by

$$\ddot{q}_j(t) + \frac{\eta p_j^2}{\rho}\dot{q}_j(t) + \frac{Ep_j^2}{\rho}q_j(t) = D\frac{Ep_j^2}{\rho}u(t), \tag{9}$$

where $u(t)$ is the voltage applied to the PEA and D is a coefficient, depending on the PEA materials used. By taking the Laplace transformation of Eq. (9), the transfer function of $Q_j(s)/U(s)$ can then be derived

$$\frac{Q_j(s)}{U(s)} = \frac{D\omega_j^2}{s^2 + 2\xi_j\omega_j s + \omega_j^2}, \tag{10}$$

where ξ_j and ω_j are the damping ratio and natural frequency of the jth-mode coordinate, and are given by $\omega_j = p_j\sqrt{E/\rho}$ and $\xi_j = (\eta p_j/2)\sqrt{1/\rho E}$, respectively.

In the present study, the displacement of the positioning mechanism driven by the PEA, denoted by $x_e(t)$ in Fig. 1, is of particular interest. It is noted that $x_e(t)$ is the same as the displacement of the end point of PEA, i.e., the motion $\upsilon(x,t)$ at $x=L$. Thus, $x_e(t)$ is also characterized by Eqs. (6) and (7) with x replaced by L, and Eq. (9) and (10). On this basis, $x_e(t)$ can be represented as the parallel second-order subsystems that are described by Eq. (10) with the numerator replaced by $D\omega_j^2\phi_j(L)$ [or $H\omega_j^2 \sin(p_j L)$, where $H=DB$]. The block diagram of the representation is shown in Fig. 2.

It is noticed from the solutions to Eqs. (7) and (8) that $\phi_j(L)$ decreases as the mode order (i.e., j) increases, depending on the ratio of the positioning mechanism mass to the PEA mass (i.e., M/m). Figure 3, for example, illustrates the magnitude ratio of the second mode to the first one as a function of the mass ratio. It can be seen that, as the mass ratio increases, the magnitude of the second mode becomes much less than that of the first one. As a result, the displacement of the positioning mechanism [i.e., $x_e(t)$] is dominated by the first mode. Thus, the second-order subsystem corresponding to the first mode can be used to approximate the dynamics of positioning system. Meanwhile the error involved in the approximation can be evaluated from other second-order subsystems, given by

$$E(s) = U(s)\sum_{j=2}^{\infty} \frac{H\omega_j^2 \sin(p_j L)}{s^2 + 2\xi_j\omega_j s + \omega_j^2}, \tag{11}$$

where $E(s)$ is the Laplace transformation of the error $e(t)$. If the mass ratio is a relatively small number, one may consider the use of two or more second-order subsystems for improved accuracy, and the error is also governed by Eq. (11) with, for example, $j=2$ replaced by $j=3$ for the use of two second-order subsystems. Therefore, for a given piezoactuated positioning system, the use of Eq. (11) allows one to rigorously choose an appropriate approximation for the system dynamics based on the desired accuracy.

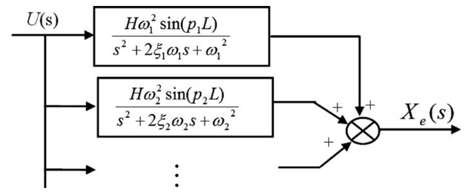

FIG. 2. Representation of the positioning mechanism displacement by using parallel second-order subsystems.

III. EXPERIMENT AND SIMULATION RESULTS

A piezoactuated positioning system was prototyped in the authors' laboratory, which is shown in Fig. 4, along with a sensor for displacement measurements. Experiments of the system step responses were designed and conducted on the

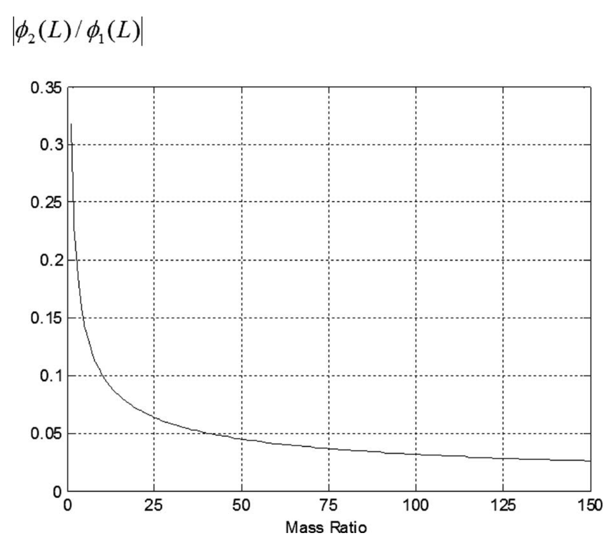

FIG. 3. $|\phi_2(L)/\phi_1(L)|$ vs the mass ratio, illustrating that the magnitude of the second mode becomes much less significant compared to the first mode as the mass ratio increases.

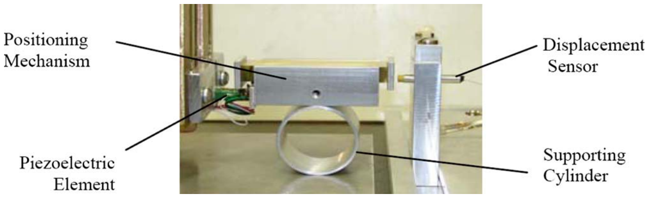

FIG. 4. (Color online) Prototype of a piezoactuated positioning system.

prototype to investigate the errors involved in the approximations of the system dynamics. In particular, step voltages of 48.2, 64.2, and 75.4 V were used to excite the positioning system with a mass ratio of 24.7, 47.7, 87.5, 115.8, and 138.8, respectively; and its displacement responses were recoded. Meanwhile, the displacements were also simulated by using different approximations. To measure the difference between the measured displacements and the model predictions, the following error index (EI) was used

$$\mathrm{EI} = \sqrt{\frac{\sum_{i=1}^{n} (\mathrm{MD}_i - \mathrm{SD}_i)^2}{(n-1)}}, \tag{12}$$

where MD is the measured displacement, SD is the simulated displacement, and n is the number of samples. A part of the results obtained in this study are shown in Figs. 5 and 6. Specifically, Fig. 5 shows the measured and simulated displacements for a mass ratio of 138.8. The solid lines represent the measured displacements, which were used for determining the parameters of the second-order system, i.e., H, ω, and ξ. Their values were found to be 5.1×10^{-7} m/V, 6736 rad/s, and 0.177. Using these values, the step responses were simulated by using one second-order approximation; and the results are shown in dash lines in Fig. 5. By using Eq. (12), the values of EI for the results shown in Fig. 5 were calculated to be 0.852×10^{-6}, 0.586×10^{-6}, and 0.381×10^{-6} m for 75.4, 64.2, and 48.2 V, respectively, illustrating the appropriateness of the use of one second-order approximation for the system dynamics for a mass ratio of 138.8. Figure 6 shows the errors involved in one second-order approximation

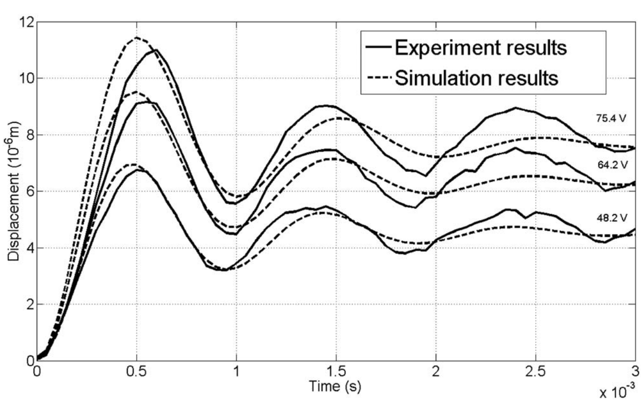

FIG. 5. Comparison of measured and simulated step responses for a mass ratio of 138.8.

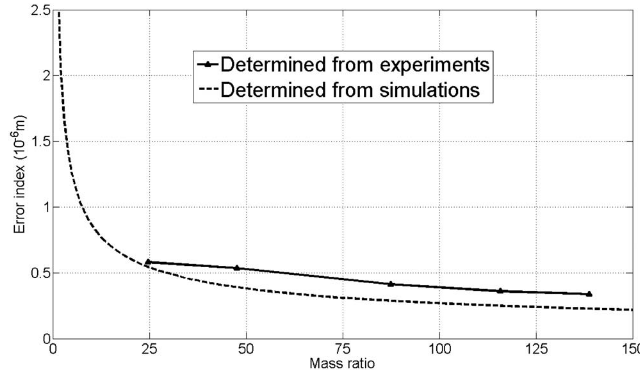

FIG. 6. Errors involved in the second-order approximations for an applied voltage of 48.2 V.

as a function of the mass ratio for an applied voltage of 48.2 V, in which the points on a solid line were determined from the measurements and the dash line was obtained based on simulations. Specifically, the points on the solid line were determined from Eq. (12), in which MD is the measured displacement and SD is the simulated displacement by means of one second-order approximation; and the dash line was also obtained from Eq. (12), but with the term of (MD-SD) evaluated from Eq. (11). A good agreement is seen between the errors determined from the experiments and the simulations, which shows the promising of Eq. (11) in the characterization of the errors associated with the approximations of the system dynamics. Besides, it is also seen that the errors decrease with the increase in the mass ratio. Most important here is that the use of Eq. (11) make it possible to quantify these errors.

IV. CONCLUSIONS

The dynamics of a piezoactuated positioning system is of importance for its design and/or control, which is approximated empirically by using a second-order system in the literature. The contribution of this note is to provide a rationale for such an approximation and a method to quantify the error associated with the approximation. The effectiveness of the method was illustrated experimentally on the prototype of a piezoactuated positioning system.

ACKNOWLEDGMENTS

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